On Dimension-Free L^p Bounds for Maximal Functions

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Contents

1 Introduction

Maximal functions are a central object in analysis that arise throughout the study of singular integral operators, PDEs, ergodic theory, and beyond. One of the simplest and most natural maximal operators is the Hardy-Littlewood maximal operator M for the Euclidean ball in $\mathbb{R}^d$. For $f : \mathbb{R}^d \to \mathbb{R}$, the Hardy-Littlewood maximal function M is defined as

$$Mf(x) = \sup_{r>0} \frac{1}{\mathrm{Vol}_n(B(x,r))} \int_{B(x,r)} |f(y)|dy,$$

the maximum of averages of f over all Euclidean balls centered at x. A classical result of this maximal function is the Hardy-Littlewood maximal inequality, from which the Lebesgue differentiation theorem follows immediately:

Theorem 1. (Hardy-Littlewood Maximal Inequality) M satisfies a weak L^1 bound with constant 3^d. Explicitly, letting μ be the Lebesgue measure, we have that

$$\mu\left(\{x \in \mathbb{R}^d : Mf(x) \geq \alpha\}\right) \leq \frac{3^d}{\alpha} \|f\|_{L^1(\mathbb{R}^d)}.$$

Corollary 1. (Lebesgue Differentiation Theorem) For any Lebesgue measurable function $f : \mathbb{R}^d \to \mathbb{R}$, we have that for almost all $x \in \mathbb{R}^d$,

$$\lim_{r\to 0} \frac{1}{\mathrm{Vol}_n(B(x,r))} \int_{B(x,r)} f(y)dy = f(x).$$

The proof of Theorem 1 follows from the Vitali covering lemma and the inner-regularity of the Lebesgue measure. We also have the much simpler observation that $\|Mf\|_{L^\infty(\mathbb{R}^d)} \leq \|f\|_{L^\infty(\mathbb{R}^d)}$, so interpolating between L^∞ and $L^{1,\infty}$ with the Marcinkiewicz interpolation theorem, we get that for all $p \in (1, \infty)$,

$$\|Mf\|_{L^p(\mathbb{R}^d)} \leq 2\left(\frac{p}{p-1}\right)^{1/p} (3^{1/p})^d \|f\|_{L^p(\mathbb{R}^d)}. \tag{1.1}$$

From this, we may conclude that the maximal function M is of *strong type (p, p)*, or $L^p(\mathbb{R}^d)$ bounded, for all d and $p \in (1, \infty]$.

A fundamental question in the study of maximal functions is the behavior of the constants $C_{p,d}$ such that

$$\|Mf\|_{L^p(\mathbb{R}^d)} \leq C_{p,d}\|f\|_{L^p(\mathbb{R}^d)}$$

We see that the Euclidean maximal function M is not bounded as an operator on $L^1(\mathbb{R}^d)$ by lower bounding $Mf(x)$ by $\|x\|^{-d}$ up to a constant, so it makes sense that as p goes to 1, the bound (1.1) goes to ∞. What is not so clear, however, is the exponential dependence on d, and whether or not this can be improved.

In 1983, Stein and Strömberg showed that Euclidean Hardy-Littlewood maximal function does in fact have $L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ operator norm bounded independent of d ([SS83]). Their proof passes through earlier work of Stein that proved the L^p boundedness of the spherical maximal function ([Ste76]), using a technique called the "method of rotations" to extend this result to dimension-independent strong (p, p) bounds for the Euclidean maximal function.

In 1986, Bourgain proved that for an *arbitrary symmetric convex body* B of volume 1, that the associated maximal operator M_B defined as

$$M_B f(x) = \sup_{r>0} \frac{1}{\mathrm{Vol}_n(x + rB)} \int_{x+rB} |f(y)|dy$$

has bounded $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ operator norm independent of dimension as well as the convex body B ([Bou86a]). Later that year, Carbery (and independently, Bourgain) extended this result to get dimension-free strong (p, p) bounds of M_B for all $p > \frac{3}{2}$ independent of B as well ([Car86], [Bou86b]).

The obstruction for this result to hold for $p \in (1, \frac{3}{2}]$ is contained in an interpolation argument central to the proof.

In 1990, Müller showed that for $p \in (1, \frac{3}{2}]$, the constants $C_{p,d}$ for the maximal operator M_B could be bounded as a function in geometric properties of B, in particular the volumes of the smallest cross sections and largest projections ([Mül90]). Using these techniques, it follows that for all $q \in [1, \infty)$ and all $p \in (1, \infty]$, we can obtain dimension-free bounds on the $L^p(\mathbb{R}^d)$ operator norms of the maximal function associated with the ℓ^q ball. Interestingly, the case of the maximal function of the ℓ^∞ ball was not settled until 2014 by Bourgain ([Bou14]), due to stark differences in the geometry of the ℓ^∞ ball compared to the other ℓ^q balls for $q \in [1, \infty)$ as the dimension grows.

Any L^p bound on M_B gives us a differentiation theorem for the convex body B (following immediately from the fact that continuous functions with compact support are dense in L^p), i.e, that for almost all $x \in \mathbb{R}^d$,

$$\lim_{r \to 0} \frac{1}{\mathrm{Vol}_d(x + rB)} \int_{x+rB} f(y)dy = f(x).$$

Independence of dimension results for a family of convex bodies in $\mathbb{R}^d$ as $d \to \infty$ can be used to prove differentiation theorems in infinite dimensional spaces. For example, Tišer showed in [Tiš88] that the independence of dimension result of the Euclidean maximal function could be used to prove a differentiation theorem for certain Gaussian measures on infinite dimensional Hilbert spaces.

The primary goal of this paper is to give a thorough exposition of many of the results discussed in the introduction, with a focus on motivation and connecting heuristics with formalism. After developing the necessary prerequisites in Section 2, we prove Stein's dimension-free L^p bounds for the Euclidean maximal function in Section 3. Section 4 is dedicated to Bourgain's dimension-independent L^2 bounds for maximal function associated to convex bodies, and Section 5 is spent expositing Carbery's approach to extending this result to $p > \frac{3}{2}$. Finally, in Section 6 we briefly outline Müller's approach to dimension-independent L^p bounds for maximal functions associated to the ℓ^q balls, for $q \in (1, \infty)$ and all $p > 1$, and discuss some open questions in the field.

1.1 A Historical Remark: Weak Type $(1, 1)$ Bounds

While some optimizations to the Vitali covering lemma can prove a slightly better weak L^1 bound for the Euclidean maximal function than the one from Theorem 1, this approach still leaves us with a bound growing exponentially in the dimension of $\mathbb{R}^d$. A significant amount of research has been done to improve these *weak type (1, 1) bounds* and understand the true behavior the $L^{1,\infty}(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ operator norm of a maximal function $C_{1,d}$ as $d \to \infty$.

In 1983, Stein and Strömberg showed that for the maximal operator associated to the Euclidean ball, $C_{1,d}$ grows linearly in the dimension d ([SS83]). In fact, the optimal constant for this linear growth was found by Melas in 2003 ([Mel03]), with constant $\frac{11+\sqrt{61}}{12}$. Stein and Strömberg also showed in the same paper that for an arbitrary symmetric convex body of volume 1, the growth of $C_{1,d}$ is bounded by $d \log d$. While we don't yet have specific results on the relation of d and $C_{1,d}$ for arbitrary ℓ^q balls, Aldaz in 2011 showed that for the cube, $C_{1,d}$ goes to infinity monotonically in d ([Ald11]).

The table below summarizes our current understanding of maximal functions:

Averaging body	Strong (p, p)	Weak $(1, 1)$
Euclidean ball	Dimension independent for $p > 1$	$Cd, C = 1.567208\ldots$
Arbitrary convex B, $\mathrm{Vol}_d(B) = 1$	Independent of d and B for $p > \frac{3}{2}$	$O(d \log d)$
ℓ^q ball, $q \in [1, \infty)$	Dimension independent for $p > 1$	$O(d \log d)$
ℓ^∞ ball	Dimension independent for $p > 1$	Goes to ∞

2 Preliminaries and Notation

We assume the reader has familiarity with Fourier Analysis and basic Harmonic analysis. For a reference, see for instance [Gra24]. Throughout this paper, when we refer to $\mathbb{R}^d$, we refer to Euclidean space endowed with the Lebesgue measure μ. We denote $\mathcal{S}(\mathbb{R}^d)$ as the set of Schwartz functions on $\mathbb{R}^d$. We also say that $X \lesssim Y$ if there is a constant C such that $X \leq CY$, and notate dependencies in C in the subscript of the inequality sign.

For $p_0 \in [1, \infty]$, we say that an operator M is weak (p_0, p_0) bounded if there exists a constant C such that

$$\|Mf\|_{L^{p_0,\infty}(\mathbb{R}^d)} \leq C\|f\|_{L^{p_0}}$$

Where the $L^{p_0,\infty}$ norm is given by

$$\|f\|_{L^{p_0,\infty}} = \inf\{C : \mu(x : f(x) > \alpha) < \frac{C^{p_0}}{\alpha^{p_0}} \text{ for all } \alpha > 0\}$$

For $p_1 \in [1, \infty]$, we say that an operator is strong (p_1, p_1) bounded, or just L^p bounded, by a constant C if our operator M has operator norm from $L^{p_1}(\mathbb{R}^d) \to L^{p_1}(\mathbb{R}^d)$ bounded by C. This operator norm will often be shorthanded as $\|M\|_{p_1 \to p_1}$.

We denote the Fourier transform of a function $f : \mathbb{R}^d \to \mathbb{R}$ as $\widehat{f}$ or $(f)^\wedge$, with

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi} d\xi.$$

We also denote the inverse Fourier transform f as $f^\vee$, defined as

$$f^\vee(x) = \int_{\mathbb{R}^d} f(-\xi)e^{-2\pi i x\cdot\xi} dx.$$

We will consider *dilations* of a function $f : \mathbb{R}^d \to \mathbb{R}$ by a constant $\lambda \in \mathbb{R}_{>0}$ as the function $x \to f(x\lambda)$, denoted by $f_{[\lambda]}$. We will also consider *normalized rescalings* of a function $f : \mathbb{R}^d \to \mathbb{R}$ by a constant $t \in \mathbb{R}_{>0}$ as the function $x \to \frac{1}{t^d} f(t^{-1}x)$, denoted by $f_{(t)}$. Note that normalized rescaling preserves the L^1 norm of f. We also note that dilation and normalized rescaling are dual under the Fourier transform, i.e.

$$(f_{(t)})^\wedge = (f)^\wedge_{[t]} \qquad (f_{[\lambda]})^\wedge = (f)^\wedge_{(\lambda)}.$$

Given a function $m : \mathbb{R}^d \to \mathbb{R}$ such that $m \in L^\infty(\mathbb{R}^d)$, we can define an L^2-bounded operator T_m by

$$Tf = (m(\xi)\widehat{f}(\xi))^\vee.$$

We call T_m a *Fourier multiplier with symbol* m. We will often refer to a Fourier multiplier only by it's symbol m, and refer to the L^p operator norm of T_m the as the L^p *multiplier norm of* m, denoted by $\|m\|_{p\to p}$. We also note the following important and simple fact that both m and m_λ have the same L^p multiplier norms.

We state a simple yet useful criterion to understand the L^p boundedness of a multiplier operator:

> **Lemma 1.** Suppose that a Fourier multiplier T with symbol $m(\xi) : \mathbb{R}^d \to \mathbb{R}$ is a bounded operator from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$. For some $\psi \in L^1(0, \infty)$, consider the Fourier multiplier N with symbol
>
> $$n(\xi) = \int_0^\infty \psi(\lambda)m(\lambda\xi)d\lambda.$$
>
> We have that
>
> $$\|n\|_{p\to p} \leq \|\psi\|_{L^1(0,\infty)}\|m\|_{p\to p}.$$

This proof follows easily from the fact that m and m_λ have the same L^p multiplier norms, since we can approximate $\psi(\lambda)$ with simple functions, use linearity of Fourier multipliers, and then apply a limiting argument.

2.1 Interpolation Results

Interpolation will play a significant role throughout this paper. Oftentimes for some $p < 2$, we will be required to bound the L^p norm of an operator. It will be much easier to work in in L^2, where we have access to Parseval's identity, which once we interpolate with a "naive" bound in L^q for $1 < q < 2$ will give us our desired result. One main tool for us will be the *Marcinkiewicz Interpolation Theorem*, which tells us that given an operator T that is weak (p_0, p_0) bounded and strong (p_1, p_1) bounded, then it is strong (p, p) bounded for all $p_0 < p < p_1$.

Theorem 2. (Marcinkiewicz Interpolation Theorem) Let $0 < p_0 < p_1 \leq \infty$, and let T be a sublinear operator on measurable functions from $\mathbb{R}^d$ to $\mathbb{R}$. If we have that T is weak (p_0, p_0) bounded by a constant A_0, and strong (p_1, p_1) bounded by a constant A_1, then for all p between p_0 and p_1, we have that T is strong (p, p) bounded by the constant

$$A = 2\left(\frac{p}{p - p_0} + \frac{p}{p_1 - p}\right)^{\frac{1}{p}} A_0^{\frac{1/p - 1/p_1}{1/p_0 - 1/p_1}} A_1^{\frac{1/p_0 - 1/p}{1/p_0 - 1/p_1}}.$$

Note that by a trivial application of Chebyschev, strong (p_0, p_0) bounded implies weak (p_0, p_0) bounded, so Marcinkiewicz can also interpolate between two strong-type bounds for a sublinear operator.

Instead of just interpolating a fixed operator between different L^p spaces, we will often want to interpolate between a family of operators as well. The result that allows us to this, *interpolation of analytic families of operators*, is a generalization of Riesz-Thorin interpolation and the setup is as follows:

Let T_z be a family of linear operators on measurable functions from $\mathbb{R}^d$ to $\mathbb{R}$ defined for all $z \in S \subset \mathbb{C}$, where S is the strip $\{z : r_0 \leq \text{Re}(z) \leq r_1\}$. Suppose that

1. The family T_z is *analytic* in the sense that for all Schwartz functions functions $f, g : \mathbb{R}^d \to \mathbb{R}$,

$$z \to \int_{\mathbb{R}^d} T_z(f(x)) \cdot g(x) dx$$

   is analytic on the the interior of S and continuous on all of S.

2. The family T_z has *admissible growth* throughout S, meaning it satisfies the growth condition that there exists an $0 < a < \pi$ such that for all $f, g \in \mathcal{S}(\mathbb{R}^d)$, there is constant $C_{f,g}$ such that for all $z \in S$

$$\log \left| \int_{\mathbb{R}} T_z(f) g \right| \leq C_{f,g} e^{a \, \text{Im}(z)}.$$

Then we have the following result:

Theorem 3. (Interpolation of analytic families of operators) Let T_z be an analytic family of linear operators satisfying the admissible growth condition above, and let $1 \leq p < q \leq \infty$. Also suppose that M_0 and M_1 are positive functions on the real line such that

$$\|T_{r_0+iy}(f)\|_{L^p} \leq M_0(y)\|f\|_{L^p}$$
$$\|T_{r_1+iy}(f)\|_{L^p} \leq M_1(y)\|f\|_{L^q}.$$

We suppose additionally that M_0 and M_1 satisfy the following growth conditions for some $0 < b < \pi$,

$$\log M_0(y), \log M_1(y) \lesssim e^{b|y|}.$$

Under these conditions, we have that for $\theta \in [0, 1]$, $p' = \frac{\theta}{p} + \frac{(1-\theta)}{q}$, and $\alpha = \theta r_0 + (1 - \theta) r_1$

$$\|T_\alpha(f)\|_{L^{p'}} \lesssim_\theta \|f\|_{L^{p'}}.$$

For proofs of both of these interpolation theorems, or their statements in full generality, see Chapter 1.3 of [Gra14].

2.2 An Introduction to Littlewood-Paley Theory

Littlewood-Paley theory is a recurring tool in the study of maximal averaging operators. In this subsection, we briefly introduce the fundamental ideas in this area and how they will be used.

We first construct a dyadic partition of unity of $\mathbb{R}_{>0}$. Fix $\psi_0 : \mathbb{R}^d \to \mathbb{R}$ to be a C^∞ function taking the value 1 when $|\xi| \le 1$ and supported on $|\xi| \le 2$. Then we define

$$\psi_j(\xi) = \psi_0(2^{-j}\xi) - \psi_0(2^{1-j}\xi)$$

Each ψ_j is supported on the annulus $\{2^{j-1} \le |\xi| \le 2^{j+1}\}$, and we have the pointwise relation that $\sum_{j\in\mathbb{Z}} \psi_j = 1$.

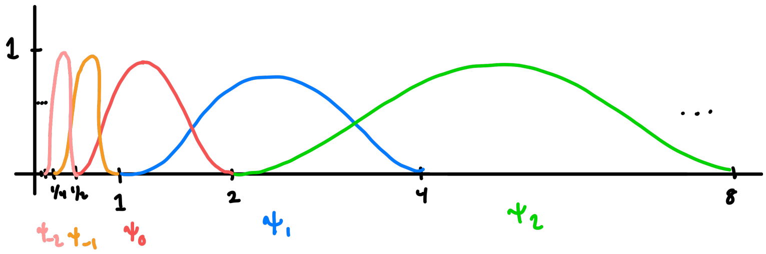

Figure 1: Our dyadic partition of unity

Using this, we define the *Littlewood-Paley projection operators* R_j as a Fourier multiplier with symbol $\psi_j(|\xi|)$, i.e.

$$\widehat{R_j f}(\xi) = \psi_j(|\xi|)\widehat{f}(\xi).$$

By Young's inequality, it is clear that R_j is L^p bounded for all p. In this way, we can write

$$f = \sum_{j\in\mathbb{Z}} R_j f.$$

What the projection operator R_j is doing is "filtering" out all frequencies not at the scale of 2^j. Then by writing f as the sum of it's projections, we are breaking f apart into functions who's frequencies are localized in a given dyadic range.

An important heuristic that comes with Littlewood-Paley theory is that functions with low frequencies relative to the interval we are studying them at are *well-behaved* in the following way: assume that we are working with a function $f : \mathbb{R} \to \mathbb{R}$ (that is, say, Schwartz) on the interval $[-1, 1]$. If we consider frequency projections $R_j f$ for $j \ll 1$, then $R_j f$ oscillates at frequencies around 2^{-j}. While this says nothing about the magnitude of $R_j f$, it does give us control on it's regularity, or how wildly it's behavior can change in this interval. Since the periods of the frequencies R_j oscillates are much larger than the interval $[-1, 1]$ we're studying $R_j f$ on, $R_j f$ cannot change it's behavior significantly in this interval, and we should expect to have a good understanding of $R_j f$ on $[-1, 1]$ for $j \ll 1$.

On the other hand, when $j \gg 1$, $R_j f$ oscillates at frequencies much finer than the width of the interval, and so we cannot expect to have a strong handle of the behavior of $R_j f$ at these scales without further tools. However, what we do have is that on this interval, $R_j f$ has approximately mean zero. The figure below illustrates both of these observations.

The analysis of maximal functions plays extremely well with this Littlewood-Paley decomposition if we break the maximal function into dyadic regions as well. Given a maximal operator M_B associated to some convex body B, we write $A_{B,r}f$ to be the average of f over the convex body $x + Br$. Then we can write

$$M_B f(x) = \sup_{r>0} A_{B,r} f(x) = \sup_{k\in\mathbb{Z}} \sup_{2^k \le r \le 2^{k+1}} A_{B,r} f(x) = \sup_{k\in\mathbb{Z}} \sup_{2^k \le r \le 2^{k+1}} \sum_{j\in\mathbb{Z}} A_{B,r} R_j f.$$

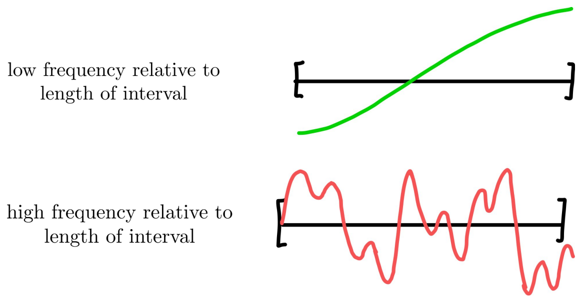

Figure 2: Littlewood-Paley heuristics

The terms $A_{B,r}R_jf$ are taking the average of some convex body at a scale of $r \sim 2^k$ for some k, of a function that oscillates at frequencies at a scale of 2^j. If j is much less than k, we have a very strong understanding of $A_{B,r}R_jf$. In the case where j is much greater than k, then since R_jf has basically mean zero on $x+rB$ for $r \sim 2^k$, $A_{B,r}R_jf$ will decay as j goes to infinity. This will rigorously be achieved by studying the structure of $A_{B,r}$ (in particular, it's a Fourier multiplier who's symbol has some nice decay properties). This phenomenon, where on a fixed interval an averaging operator picks up higher and higher frequencies less and less, is an instance of a general phenomenon of *almost orthogonality* in Littlewood-Paley theory, where if a function's frequency is localized away from where an operator is "looking," it's contribution is negligible.

3 Dimension-Free L^p Bounds for the Euclidean Ball Maximal Function

Throughout this section, we assume $B(x,r)$ to be the Euclidean ball centered at x and of radius r. One of the most foundational results in the study of maximal functions is that for $p \in (1,\infty)$, the Hardy-Littlewood maximal function for the Euclidean ball

$$Mf(x) = \sup_{r>0} \frac{1}{\mathrm{Vol}_d\, B(x,r)} \int_{B(x,r)} |f(y)|dy,$$

has $L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ operator norm bounded independent of the dimension d. In this section, we will exposit a proof of Stein's original proof of this result ([Ste82] and [SS83]), with inspiration from two other expositions, [Alm19] and [Tao11]. Our proof will first pass through a proof that the spherical maximal function M_S has bounded $L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ operator norm, where $M_Sf(x)$ is defined to be the maximum of all spherical averages of f centered at x:

$$M_Sf(x) = \sup_{r>0} \int_{S^{d-1}} |f(x+r\omega)|d\sigma^{d-1}(\omega). \tag{3.1}$$

In the above equation, $d\sigma^{d-1}$ denotes the normalized surface measure on S^{d-1}. Then, we will use a technique called *the method of rotations* to show that our resulting operator norm bound can be made independent of d. Since pointwise,

$$Mf(x) \leq M_Sf(x)$$

we will conclude the result.

3.1 Boundedness of the Spherical Maximal Function

We first study the spherical maximal function (3.1) on Schwartz functions $f : \mathbb{R}^d \to \mathbb{R}$. We write M_Sf as

$$M_Sf = \sup_{r>0} A_r|f|,$$

where we denote A_rf as the spherical average

$$A_rf = \int_{S^{d-1}} f(x-r\omega)d\sigma^{d-1}(\omega).$$

For notational convenience, we will drop the exponent on $d\sigma^{d-1}$ when it is clear what measure we are working with. Our main goal will be to prove a statement about the L^p boundedness of M_S:

Theorem 4. (Stein's Spherical Maximal Theorem) Let $d \geq 3$. Then for each $\frac{d}{d-1} < p \leq \infty$, and $f \in \mathcal{S}(\mathbb{R}^d)$, we have that

$$\|M_S f\|_{L^p(\mathbb{R}^d)} \leq C_{d,p}\|f\|_{L^p(\mathbb{R}^d)}$$

for $C_{d,p}$ some constant dependent on d and p.

Throughout this subsection, when we write inequalities $\lesssim$, the constants in these inequalities are dependent on p and d.

To prove this, we first note that M_S has bounded operator norm from L^∞ to L^∞, since we have the pointwise bound $M_S f(x) \leq \|f\|_\infty$. Therefore, if we prove Stein's spherical maximal theorem for all $\frac{d}{d-1} < p \leq 2$, then we can interpolate with the (∞, ∞) bound to conclude the result for the p we desire.

We will first study a localized version M_S^1 of the maximal operator M_S, where we only average over spheres between radii 1 and 2:

$$M_S^1 = \sup_{1 \leq r \leq 2} A_r|f|.$$

We will see that the argument for M_S^1 will generalize to all of M_S. We make use of Littlewood-Paley theory and decompose our function f using an annular frequency decomposition. Just as in the preliminaries, we fix $\psi_0 : \mathbb{R}^d \to \mathbb{R}$ to be a C^∞ function taking the value 1 when $|\xi| \leq 1$ and supported on $|\xi| \leq 2$, and define $\psi_k(\xi) = \psi_0(2^{-k}\xi) - \psi_0(2^{1-k}\xi)$, supported on the annulus $\{2^{k-1} \leq |\xi| \leq 2^{k+1}\}$. Pointwise, this satisfies $\sum_{j\in\mathbb{Z}} \psi_j = 1$.

Now we decompose f into it's frequency projections $R_j f$, where $\widehat{R_j f}(\xi) = \psi_j(\xi)\widehat{f}(\xi)$. Since f is Schwartz, we can use the inverse Fourier transform to write

$$f = \sum_{k\in\mathbb{Z}} R_k f.$$

To bound M_S^1, we split f into it's "low" and "high" frequencies. We let $R_{\leq 1} f = \sum_{k \leq 1} R_k f$. Then by the triangle inequality and the subadditivity of M_S, proving the L^p boundedness of M_S^1 reduces to proving the following two claims:

$$\|M_S^1(R_{\leq 1} f)\|_{L^p(\mathbb{R}^d)} \leq C_{d,p}\|f\|_{L^p(\mathbb{R}^d)} \tag{3.2}$$

$$\|M_S^1(R_k f)\|_{L^p(\mathbb{R}^d)} \leq C_{d,p,k}\|f\|_{L^p(\mathbb{R}^d)} \text{ for } k > 1, \text{ and } \sum_{k=2}^{\infty} C_{d,p,k} < \infty \tag{3.3}$$

The motivation behind this decomposition is that that the operator M_S^1 averages over radii at a dyadic scale of 1 (so $1 \leq r \leq 2$). The frequency projections $R_k f$ for $k \leq 1$ cannot change at finer frequencies than $|\xi| \leq 1$. In this way, we expect for $M_S^1 R_j f(x)$ to be approximately $f(x)$, since the behavior of $R_j f(x)$ for $j \leq 1$ cannot change much in this annulus. When $k > 1$, the frequencies R_k oscillates at has period much smaller than the width of the annulus that we are studying, and so has approximately mean zero on the annulus. Therefore, we expect $M_S^1 R_j f$ to be small at these scales.

3.1.1 Proving the first claim

To prove (3.2), we note that we can write $R_j f$ as a convolution of f with the Schwartz function $\widehat{\psi}_j$. Therefore we can write $R_{\leq 1} f$ as $f * \varphi$, where $\varphi = \sum_{j\leq 1} \widehat{\psi}_j$. By Fubini's theorem, we can write that

$$\begin{aligned} A_r R_{\leq 1} f(x) = A_r(\varphi * f)(x) &= \int_{S^{d-1}} \int_{\mathbb{R}^d} f(y)\varphi((x - r\omega) - y) dy\, d\sigma(\omega) \\ &= \int_{\mathbb{R}^d} f(y) \int_{S^{d-1}} \varphi((x - y) - r\omega) d\sigma(\omega)\, dy = f * A_r\varphi. \end{aligned}$$

Now since $\varphi(y)$ is Schwartz (in fact, compactly supported), it is bounded up to a constant by $\frac{1}{(1+|y|)^{100d}}$ (this constant is dependent on d since the ψ_j are). Thus, we can bound $A_r\varphi$ for $r \in [1, 2]$ by

$$|A_r\varphi(x)| \le \int_{S^{d-1}} |\varphi(x-r\omega)|d\sigma(\omega) \lesssim \int_{S^{d-1}} \frac{1}{(1+|x-r\omega|)^{100d}}d\sigma(\omega) \lesssim \frac{1}{(1+|x|)^{100d}}.$$

With this, we get the pointwise bound

$$|A_r R_{\le 1} f(x)| \le |f * A_r\varphi| \lesssim \int_{\mathbb{R}^d} |f(x-y)|\frac{dy}{(1+|y|)^{100d}}.$$

Heuristically, this just looks like averaging f on a small ball around x, so we should have that

$$\int_{\mathbb{R}^d} |f(x-y)|\frac{dy}{(1+|y|)^{100d}} \lesssim Mf(x).$$

This easily follows from the following calculation:

$$\int_{\mathbb{R}^d} |f(x-y)|\frac{dy}{(1+|y|)^{100d}} = \sum_{k\in\mathbb{N}} \int_{k\le|y|\le k+1} |f(x-y)|\frac{dy}{(1+|y|)^{100d}}$$
$$\lesssim \sum_{k\in\mathbb{N}} k^{-100d} \int_{k\le|y|\le k+1} |f(x-y)|dy \lesssim \sum_{k\in\mathbb{N}} k^{-100d}|B(x,k)|Mf(x)$$
$$\lesssim \sum_{k\in\mathbb{N}} k^{-100d}k^d Mf(x) \lesssim Mf(x).$$

Since Mf is L^p bounded for $p \in (1,\infty)$, as shown in the introduction, the fact that $|A_r R_{\le 1} f(x)|$ is pointwise bounded by Mf uniformly for $1 \le r \le 2$ allow us to conclude that $|A_r R_{\le 1} f(x)|$ is L^p bounded as well, proving (3.2).

3.1.2 Proving the second claim

We now work toward (3.3). If we run the same approach we took with the first claim for every $k > 1$, we again can write $A_r R_k f(x) = f * A_r\varphi_k$, where $\varphi_k = \widehat{\psi}_k$. However, the constant such that

$$A_r\varphi_k(x) \lesssim \frac{1}{(1+|x|)^{100d}}$$

is dependent on k exponentially, which intuitively can be seen by noting that can be seen noting $\widehat{\varphi}_k = (x-r\omega) = 2^{kd}\widehat{\psi}_0(2^k\xi) - 2^{(k-1)d}\widehat{\psi}_0(2^{k-1}\xi)$ is an approximation to the identity of height 2^{kd} and width 2^{-k}. Formally, we have that

$$|A_r\varphi_k(x)| = \int_{S^{d-1}} |\varphi_k(x-r\omega)|d\sigma(\omega)$$
$$\le 2^{kd}\int_{S^{d-1}} |\widehat{\psi}_0(2^k(x-r\omega))| + |2^{-d}\widehat{\psi}_0(2^{k-1}(x-r\omega))|d\sigma(\omega) \lesssim 2^k(1+|x|)^{-100d}.$$

With this, we see that

$$|A_r R_k f(x)| = |f * A_r\widehat{\varphi_k}(x)| \lesssim \int_{\mathbb{R}^d} |f(x-y)|\frac{2^k}{(1+|y|)^{100d}}dy \lesssim 2^k Mf(x),$$

and taking supremums on both sides gives us the pointwise bound

$$\|M_S^1 R_k f\|_{L^p} \lesssim 2^k\|Mf(x)\|_{L^p} \lesssim 2^k\|f\|_{L^p} \tag{3.4}$$

for all $p \in (1,\infty)$. While this bound isn't good enough to prove (3.3) on it's own, since $2^k \to \infty$ as $k \to \infty$, we can make use of it by interpolating it with another result. In the regime where $k > 1$, as mentioned before we need to make use of the fact that A_r is a Fourier multiplier who's symbol has rapid decay. Precisely, we have that

$$\widehat{A_r f}(\xi) = \int_{\mathbb{R}^d}\int_{S^{d-1}} f(x-r\omega)d\sigma(\omega)\, e^{-2\pi i x\cdot\xi}dx$$
$$= \int_{\mathbb{R}^d}\int_{S^{d-1}} f(x)e^{-2\pi i x\cdot\xi}e^{-2\pi i r\omega\cdot\xi}d\sigma(\omega)dx = \widehat{d\sigma}(r\xi)\widehat{f}(\xi).$$

So A_r has the symbol $\widehat{d\sigma}(r\xi) = \int_{S^{d-1}} e^{-2\pi i\omega\cdot r\xi} d\sigma(\omega)$. Furthermore, a well known fact on estimates of Bessel functions (see the appendices of [Gra14] for details) tells us that

$$|\widehat{d\sigma}(\xi)|,\ |\nabla\widehat{d\sigma}(\xi)| \leq \frac{C_d}{(1+|\xi|)^{\frac{d-1}{2}}}. \tag{3.5}$$

Due to the rapid decay of $|\widehat{d\sigma}(\xi)|$ and $|\nabla\widehat{d\sigma}(\xi)|$ as ξ gets large, we are motivated to study L^2 bounds of $M_S R_k f$ for $k > 1$, where we have access to Plancharel, and then interpolate this bound with (3.4).

Using the fact that $\widehat{R_k f}$ is supported on annulus of inner radius 2^{k-1} and outer radius 2^{k+1}, we have that for $r \in [1,2]$,

$$\begin{aligned}\|A_r R_k f\|_{L^2} = \|\widehat{A_r R_k f}\|_{L^2} = \|\widehat{R_k f}(\xi)\widehat{d\sigma}(r\xi)\|_{L^2} &\lesssim 2^{-k(d-1)/2}\|\widehat{R_k f}\| \\ &\leq 2^{-k(d-1)/2}\|\widehat{f}\|_{L^2} \leq 2^{-k(d-1)/2}\|f\|_{L^2}.\end{aligned} \tag{3.6}$$

This isn't strong enough for us, since we need to take an uncountable supremum over r (otherwise, we could bound an supremum in r with a sum or a square sum in r). However, the Littlewood-Paley philosophy tell us that on dyadic scales of 2^j, we expect our function to be controlled in oscillation on intervals of size 2^{-j}. In this way, we work to change the supremum over $[1,2]$ into a maximum of a discrete set of points spaced approximately 2^{-k} away from each other (since intuitively, the function shouldn't "change much" between these points, so this countable maximum should capture all the information we need). Heuristically, what we expect to happen is if we discretize our sum as described,

$$\sup_{1\leq r\leq 2} |A_r R_k f(x)| \approx \sup_{\substack{n\in\mathbb{N} \\ 1\leq 2^{-k}n\leq 2}} |A_n R_k f(x)|$$

then approximate this finite supremum with the square function and take L^2 norms, we get that

$$\begin{aligned}\|\sup_{1\leq r\leq 2} |A_r R_k f|\|_{L^2} \lesssim \|(\sum_{\substack{n\in\mathbb{N} \\ 1\leq 2^{-k}n\leq 2}} |A_n R_k f|^2)^{\frac{1}{2}}\|_{L^2} &\lesssim \sum_{\substack{n\in\mathbb{N} \\ 1\leq 2^{-k}n\leq 2}} \|A_n R_k f\|_{L^2} \\ &\lesssim \left(2^k (2^{-k(d-1)/2}\|f\|_{L^2})^2\right)^{\frac{1}{2}} = 2^{-k(d-2)/2}\|f\|_{L^2}.\end{aligned}$$

We now work to achieve this bound rigorously. We consider a 2^{-k}-net $\{t_\tau\}_{\tau\leq 2^k}$ of $[1,2]$, which is a set of 2^k points $\{t_1, t_2, \dots, t_{2^j}\}$ where t_τ and $t_{\tau+1}$ are spaced out on the order of 2^{-k} from each other. The fundamental theorem of calculus and the triangle inequality tell us that for any C^1 function ϕ and two real numbers $s_1 < s_2$,

$$\sup_{s_1\leq t\leq s_2} \phi(t) \leq |\phi(s_1)| + \int_{s_1}^{s_2} |\phi'(s)| ds. \tag{3.7}$$

Plugging in $A_t R_k f$ for ϕ and $t_\tau, t_{\tau+1}$ for s_1, s_2 in the above identity, and taking L^2 norms, we get that

$$\left\| \sup_{t_\tau\leq t\leq t_{t_{\tau+1}}} |A_t R_k f| \right\|_{L^2} \leq \|A_{t_\tau} R_k f\|_{L^2} + \left\| \int_{t_\tau}^{t_{\tau+1}} \left| \frac{d}{dr} A_r R_k f \right| dr \right\|_{L^2}. \tag{3.8}$$

We already know the contribution of the first term on the right by (3.6), since $t_\tau \in [1,2]$. We now bound the second term on the right. Since $A_r R_k f$'s derivative in r is L^1, by the mean value theorem and the Lebesgue dominated convergence theorem we can pull the derivative in r out of the Fourier transform and calculate

$$\left(\frac{d}{dr} A_r R_k f\right)^\wedge (\xi) = \frac{d}{dr}(A_r R_k f)^\wedge(\xi) = \frac{d}{dr}\widehat{d\sigma}(\xi)\widehat{R_k f}(\xi) = \xi \cdot \nabla\widehat{d\sigma}(r\xi)\widehat{R_k f}(\xi).$$

Now we have by Minkowski's inequality for integrals that

$$\begin{aligned}\left\| \int_{t_\tau}^{t_{\tau+1}} \left| \frac{d}{dr} A_r R_k f(\cdot) \right| dr \right\|_{L^2} &\leq \int_{t_\tau}^{t_{\tau+1}} \left\| \frac{d}{dr} A_r R_k f(\cdot) \right\|_{L^2} dr \\ &= \int_{t_\tau}^{t_{\tau+1}} \|\xi \cdot \nabla\widehat{d\sigma}(\xi)\widehat{R_k f}(r\xi)\|_{L^2} dr.\end{aligned}$$

Using the fact that $\widehat{R_k f}$ is supported on $[2^{k-1}, 2^{k+1}]$, as well as the decay of $\nabla \widehat{d\sigma}$ from (3.5), we have that $\xi \cdot \nabla \widehat{d\sigma}(r\xi)$ for $1 \le r \le 2$ is maximized by $\frac{2^k}{(2^k r)^{\frac{d-1}{2}}} \le 2^{-k(\frac{d-3}{2})}$. Also using that $|\xi \cdot \nabla \widehat{d\sigma}(r\xi)| \le |\xi||\nabla \widehat{d\sigma}(r\xi)|$,

$$\|\xi \cdot \nabla \widehat{d\sigma}(\xi) \widehat{R_k f}(r\xi)\|_{L^2} \le 2^{-k(\frac{d-3}{2})} \|f\|_{L^2},$$

allowing us to conclude that

$$\left\| \int_{t_\tau}^{t_{\tau+1}} \left| \frac{d}{dr} A_r R_k f(\cdot) \right| dr \right\|_{L^2} \lesssim (t_{\tau+1} - t_\tau) 2^{-k(\frac{d-3}{2})} \|f\|_{L^2} \lesssim 2^{-k(\frac{d-1}{2})} \|f\|_{L^2}. \tag{3.9}$$

With this, we use (3.8) to carry out the bound $\|\sup_{1 \le t \le 2} |A_r R_k f|\|_{L^2}$ we have worked toward. We see that

$$\| \sup_{1 \le t \le 2} |A_r R_k f| \|_{L^2} = \| \sup_{\tau} \sup_{t_\tau \le t \le t_{\tau+1}} |A_r R_k f| \|_{L^2} \le \left(\sum_{\tau} \| \sup_{t_\tau \le t \le t_{\tau+1}} |A_r R_k f| \|_{L^2}^2 \right)^{\frac{1}{2}}.$$

Plugging in (3.8) with (3.6) and (3.9), as well as using the fact that $(a+b)^2 \lesssim a^2 + b^2$, gives us

$$\| \sup_{1 \le t \le 2} |A_r R_k f| \|_{L^2} \lesssim \left(\sum_{\tau} 2 \cdot (2^{-k(\frac{d-1}{2})})^2 \|f\|_{L^2}^2 \right)^{\frac{1}{2}} \lesssim 2^{-k(\frac{d-2}{2})} \|f\|_{L^2}. \tag{3.10}$$

Note that working over intervals of length $t_{\tau+1} - t_\tau$ is what made the contribution of the derivative in (3.9) comparable in size to the contribution from (3.6), confirming our Littlewood-Paley heuristic that $A_r R_k f$ is controlled over our net. Working over the interval $[1,2]$ instead of our net would have given us a worse bound on $\|\sup_{1 \le t \le 2} |A_r R_k f|\|_{L^2}$ by a factor of $2^{\frac{k}{2}}$.

Interpolating our two bounds for $M_S^1 R_k$, the $L^2 \to L^2$ bound (3.10) and the $L^q \to L^q$ bound (3.4) for some $q = 1 + \varepsilon$, we get by Marcinkiewicz that the the $L^p \to L^p$ operator norm of $M_S^1 R_k$ is bounded by

$$2 \left(\frac{p}{p - (1+\varepsilon)} + \frac{p}{2-p} \right)^{\frac{1}{p}} \cdot 2^{-k(-\frac{d}{p} + d - 1) + \frac{k\varepsilon d}{2}} \lesssim 2^{-k(-\frac{d}{p} + d - 1 - \frac{\varepsilon d}{2})}.$$

If we take $p > \frac{d}{d-1}$ and ε small enough, we see that the exponent above is negative, and so the sum

$$\sum_{k=2}^{\infty} 2^{-k(-\frac{d}{p} + d - 1 - \frac{\varepsilon d}{2})}.$$

converges. Therefore, we have proved (3.3), completing the proof that M_S^1 is $L^p \to L^p$ bounded for all $p > \frac{d}{d-1}$.

3.1.3 The general case

Now we work with the entire spherical maximal function M_S, which we can write as

$$M_S f = \sup_{j \in \mathbb{Z}} \sup_{2^j \le r \le 2^{j+1}} |A_r f|.$$

For every $j \in \mathbb{Z}$, we split f into its low and high frequency parts relative to the scale of the annuli the operator $\sup_{2^j \le r \le 2^{j+1}} |A_r f|$ is averaging over. Letting $R_{\le -j} f = \sum_{k < -j} R_k f$, to show that $M_S f$ has bounded $L^p \to L^p$ norm, it suffices to show (by the triangle inequality and subadditivity of M_S) the following two bounds:

$$\| \sup_j \sup_{2^j \le r \le 2^{j+1}} |A_r R_{\le -j} f| \| \le C_{d,p} \|f\|_{L^p(\mathbb{R}^d)} \tag{3.11}$$

$$\| \sup_j \sup_{2^j \le r \le 2^{j+1}} |A_r R_{-j+k} f| \|_{L^p(\mathbb{R}^d)} \le C_{d,p,k} \|f\|_{L^p(\mathbb{R}^d)} \text{ for } k \in \mathbb{N}, \text{ and } \sum_{j=1}^{\infty} C_{d,p,k} < \infty \tag{3.12}$$

Note that (3.11) bounds the parts of the Littlewood-Paley decomposition of $A_r f$ that oscillate slowly with respect to the averaging radius, and (3.12) bounds the parts that oscillate quickly with respect to the averaging radius. The claim of (3.12), that as k grows much larger than j, the operator norm of $\|\sup_j \sup_{2^j\le r\le 2^{j+1}} |A_r R_{-j+k} f|\|_{L^p(\mathbb{R}^d)}$ decays sufficiently fast, is another instance of the phenomenon of "almost orthogonality" discussed previously.

Proving (3.11) goes exactly the same as proving (3.2): $A_r R_{\le -j} f$ can be written as the convolution of f with $A_r\varphi$ for a Schwartz function φ independent of f, and by the exact same calculations as earlier we conclude the pointwise bound

$$|A_r R_{\le -j} f(x)| \lesssim Mf(x).$$

This bound holds independently of $j\in\mathbb{Z}$ and $r\in[2^j,2^{j+1}]$, which immediately proves (3.11).

To prove (3.12), we first note that by the exact same argument as in the case of M^1_S, we have that for $2^j\le r\le 2^{j+1}$,

$$|A_r R_{-j+k} f(x)| \lesssim 2^k Mf(x),$$

where this constant is independent of r and j. Therefore, we have that for all $p\in(1,\infty)$,

$$\|\sup_j \sup_{2^j\le r\le 2^{j+1}} |A_r R_{-j+k} f(x)|\|_{L^p} \le 2^k\|f\|_{L^p}. \tag{3.13}$$

We now need an L^2 bound to interpolate against. To do this, we will bound $\|\sup_{2^j\le r\le 2^{j+1}} |A_r R_{-j+k} f(x)|\|_{L^2}$ and then square sum over j. Again, $A_r R_{-j+k}$ can't oscillate very finely at scales of 2^{j-k}, so we discretize the supremum $\sup_{2^k\le r\le 2^{k+1}}$ using a 2^{j-k}-net $\{t_\tau\}_\tau < 2^{j-k}$ (again, a set of points $\{t_\tau\}_{\tau<2^k}$ in $[2^j,2^{j-1}]$ spaced out on the order of 2^{j-k}). Using the fundamental theorem of calculus identity (3.7) from earlier, plugging in $\phi(r)=A_rR_{-j+k}f(x)$, and taking L^2 norms, we get that

$$\left\|\sup_{t_\tau\le t\le t_{\tau+1}} |A_t R_{-j+k} f|\right\|_{L^2} \lesssim \|A_{t_\tau} R_{-j+k} f\|_{L^2} + \left\|\int_{t_\tau}^{t_{\tau+1}} \left|\frac{d}{dr}A_r R_{-j+k} f\right| dr\right\|_{L^2}. \tag{3.14}$$

For $2^j\le r\le 2^{j+1}$, by (3.5) and the fact that $\widehat{R_{-j+k}}f(\xi)$ is supported on $[2^{-j+k-1},2^{-j+k+1}]$, we have that

$$\begin{aligned}\|A_r R_{-j+k} f\|_{L^2} &= \|A_r\widehat{R_{-j+k}}f\|_{L^2} = \|\widehat{R_{-j+k}}f(\xi)\widehat{d\sigma}(r\xi)\|_{L^2}\\ &\lesssim 2^{-k(d-1)/2}\|\widehat{R_{-j+k}}f\|_{L^2} \le 2^{-k(d-1)/2}\|R_{-j+k}f\|_{L^2}.\end{aligned} \tag{3.15}$$

Since $(\frac{d}{dr}A_rR_{-j+k}f(\xi))^\wedge = \xi\cdot\nabla\widehat{d\sigma}(r\xi)\widehat{R_{-j+k}}f(\xi)$, using (3.5) and the fact that $|\xi\cdot\nabla\widehat{d\sigma}(r\xi)|\le|\xi||\nabla\widehat{d\sigma}(r\xi)|$, we get that for $2^j\le r\le 2^{j+1}$,

$$\begin{aligned}\|\frac{d}{dr}A_rR_{-j+k}f\|_{L^2} &= \|(\frac{d}{dr}A_rR_{-j+k}f)^\wedge\|_{L^2} = \|\xi\cdot\nabla\widehat{d\sigma}(r\xi)\widehat{R_{-j+k}}f(\xi)\|_{L^2}\\ &\lesssim 2^{-j}2^{-k(d-3)/2}\|\widehat{R_{-j+k}}f\|_{L^2} \le 2^{-j}2^{-k(d-3)/2}\|R_{-j+k}f\|_{L^2}.\end{aligned}$$

Therefore, we have the estimate that for $t_\tau,t_{\tau+1}$ in our 2^{j-k} net of $[2^j,2^{j+1}]$,

$$\left\|\int_{t_\tau}^{t_{\tau+1}} |\frac{d}{dr}A_rR_{-j+k}f(\cdot)|dr\right\|_{L^2} \lesssim (t_{\tau+1}-t_\tau)2^{-j}2^{-k(\frac{d-3}{2})}\|R_{-j+k}f\|_{L^2} \lesssim 2^{-k(\frac{d-1}{2})}\|R_{-j+k}f\|_{L^2}. \tag{3.16}$$

Plugging (3.15) and (3.16) into (3.14) gives us that

$$\left\|\sup_{t_\tau\le t\le t_{\tau+1}} |A_tR_{-j+k}f|\right\|_{L^2} \lesssim 2^{-k(\frac{d-2}{2})}\|R_{-j+k}f\|_{L^2},$$

and with this we can calculate that

$$\begin{aligned}\left\|\sup_{2^j\le t\le 2^{j+1}} A_tR_{-j+k}f\right\|_{L^2} &= \left\|\sup_\tau \sup_{t_\tau\le t\le t_{\tau+1}} A_tR_{-j+k}f\right\|_{L^2} \le \left(\sum_\tau \|\sup_{t_\tau\le t\le t^{\tau+1}} A_tR_{-j+k}f\|_{L^2}^2\right)^{1/2}\\ &\lesssim \left(\sum_\tau 2^{-k(d-1)}\|R_{-j+k}f\|_{L^2}\right)^{1/2} \lesssim 2^{-k(\frac{d-2}{2})}\|R_{-j+k}f\|_{L^2}.\end{aligned} \tag{3.17}$$

Finally, we can bound $\|\sup_j \sup_{2^j\le r\le 2^{j+1}} |A_r R_{-j+k} f(x)|\|_{L^2}$ by estimating a supremum in j by a square sum in j, and using (3.17):

$$\begin{aligned}\|\sup_{j\in\mathbb{Z}} \sup_{2^j\le r\le 2^{j+1}} |A_r R_{-j+k} f|\|_{L^2} &\le \left(\sum_{j\in\mathbb{Z}} \| \sup_{2^j\le r\le 2^{j+1}} |A_r R_{-j+k} f(\cdot)|\|_{L^2}^2\right)^{\frac{1}{2}} \\ &\lesssim \left(\sum_{j\in\mathbb{Z}} 2^{-k(d-2)}\|R_{-j+k} f\|_{L^2}^2\right)^{\frac{1}{2}} \lesssim 2^{-k(\frac{d-2}{2})}\|f\|_{L^2}. \end{aligned} \tag{3.18}$$

In the last equality, we used the fact that $\left(\sum_{k\in\mathbb{Z}}\|R_{-j+k} f\|_{L^2}\right)^{\frac{1}{2}} = \|f\|_{L^2}$, which follows from Parseval below:

$$\left(\sum_{k\in\mathbb{Z}}\|R_{-j+k} f\|_{L^2}^2\right)^{\frac{1}{2}} = \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 \sum_{k\in\mathbb{Z}} \psi_{-j+k}(|\xi|)^2 d\xi.$$

Since ψ_{-j+k} is supported in $[2^{-j+k-1}, 2^{-j+k+1}]$, the sum $\sum_{k\in\mathbb{Z}} \psi_{-j+k}(|\xi|)^2$ is at most 2 for any $\xi \in \mathbb{R}^d$. One final application of Parseval gives the desired inequality[1].

Interpolating (3.18) with (3.13) using Marcinkiewicz gives us again, that for $p > \frac{d}{d+1}$, M_S is $L^p \to L^p$ bounded. This completes the proof of Theorem 3.1.

3.2 The Calderón-Zygmund Method of Rotations

Now that we have shown that the spherical maximal function is L^p bounded on Schwartz functions for $p > \frac{d}{d+1}$, i.e. for $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\|M_S f(x)\|_{L^p(\mathbb{R}^d)} \le C_{d,p}\|f\|_{L^p(\mathbb{R}^d)},$$

we will show that this constant is in fact independent of dimension d.

Before discussing this argument, we show why independence of the spherical maximal function implies independence of dimension for the maximal function for the Euclidean ball. Intuitively, these two maximal functions are related: If the average of $|f|$ on $B(x,r)$ is some value K, then the average value of $|f|$ on some sphere centered at x and of radius less than r must be at least K. Therefore, we expect to have the pointwise bound

$$Mf(x) \le M_S f(x) \tag{3.19}$$

This is seen rigorously by switching to polar coordinates, where ω_d denotes the surface area of S^d.

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|dy = \frac{1}{|B(x,r)|}\int_0^r \omega_d r^{n-1} A_r f(x) dr \le M_S f(x) \frac{\int_0^r \omega_d r^{n-1}}{|B(x,r)|} = M_S f(x).$$

Taking supremum over r on both sides gives (3.19). Therefore, we will have shown that for a fixed p and for all Schwartz functions $f : \mathbb{R}^d \to \mathbb{R}$, $\|Mf\|_{L^p(\mathbb{R}^d)} \le K\|f\|_{L^p(\mathbb{R}^d)}$ for some constant K independent of d satisfying $p > \frac{d}{d+1}$. Combining this with the fact that the $L^p(\mathbb{R}^d)$ operator norm of Mf is bounded for the finite number of d satisfying $p < \frac{d}{d+1}$(as shown by the Vitali covering argument in the introduction) shows that an L^p bound for the Euclidean maximal function on Schwartz functions $f : \mathbb{R}^d \to \mathbb{R}$ can be taken independent of d.

We still need to get rid of the Schwartz condition which we needed to use Fourier analysis techniques on the spherical maximal function. This is easily disposed of using a standard density argument and the subadditivity of Mf. Since M is a sublinear bounded operator from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$, we have that for a sequence of measurable functions f_n converging in L^p to a function f, $\|Mf_n\|_{L^p} \to \|Mf\|_{L^p}$. Now

[1]This is one direction of the L^2 case of the general Littlewood-Paley inequalities, which states that for any $p \in (1,\infty)$, the L^p norm of the square function $\left(\sum_{j\in\mathbb{Z}}(P_j f)^2\right)^{1/2}$ is up to some constant factor equal to the L^p norm of f. Thus, Littlewood-Paley theory gives some sense of "orthogonality" that one would a-priori only expect to see in the L^2 world.

taking f to be an arbitrary L^p function, and f_n to be a sequence of Schwartz functions approximating it in L^p norm (since Schwartz functions are dense in L^p), taking limits on both sides of $\|Mf_n\|_{L^p(\mathbb{R}^d)} \leq K_p\|f_n\|_{L^p(\mathbb{R}^d)}$ allows us to conclude that the Euclidean maximal function is bounded independent of dimension.

We now proceed to prove that

Theorem 5. For $p > \frac{d}{d+1}$, the constants $C_{p,d}$ in the L^p bounds of the spherical maximal function can be taken to be independent of d.

From the boundedness of the spherical maximal function, for $f : \mathbb{R}^d \to \mathbb{R}$ we have that $\|M_S f\|_{L^p(\mathbb{R}^d)} \leq C(d,p)\|f\|_{L^p(\mathbb{R}^d)}$, for some constant $C(d,p)$ depending on d and p. We would like to find a relation between $C(d,p)$ and $C(d+1,p)$, namely, show that they can be made equal. To do so, we need a way to relate the spherical maximal function on $\mathbb{R}^d$ to the spherical maximal function on $\mathbb{R}^{d+1}$. A natural way to approach this is through the method of rotations: viewing an average over the sphere S^d as taking averages of all copies of S^{d-1} in S^d, and then averaging over these values.

In this manner, for $w_0 \in S^d \subset \mathbb{R}^{d+1}$, we first let $U_{\omega_0} \in SO_{d+1}$ be an orthogonal transformation that takes the dth standard basis vector $e_{d+1} \in \mathbb{R}^{d+1}$ to ω_0. More specifically, with the canonical embedding of $\mathbb{R}^d$ in $\mathbb{R}^{d+1}$, U_{ω_0} is an orthogonal transformation that maps $S^{d-1} \subset \mathbb{R}^d$ to the set $\{x \in S^d : x \perp \omega_0\}$.

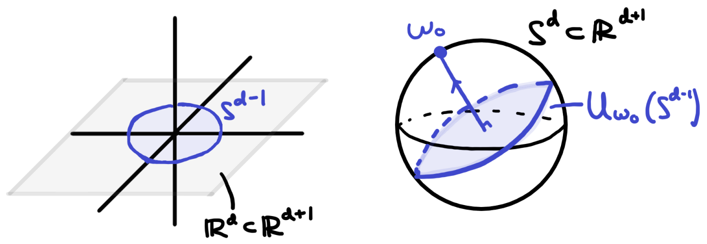

Figure 3: Viewing S^{d-1} as a subset of S^d

We create a maximal function over S^{d-1}s about x in the plane $U_{\omega_0}(\mathbb{R}^d) \subset \mathbb{R}^{d+1}$ as

$$M_S^{\omega_0} f(x) := \sup_{r>0} A_r^{\omega_0}|f|(x),$$

where $A_r^{\omega_0}$ is defined as

$$A_r^{\omega_0} f(x) := \int_{S^{d-1}} f(x - rU_{\omega_0}\omega) d\sigma^{d-1}(\omega).$$

We note that if $\omega_0 = e_{d+1}$, then we can easily compute using Fubini's theorem that for $f : \mathbb{R}^{d+1} \to \mathbb{R}$,

$$\begin{aligned}
\|M_S^{e_{d+1}} f\|_{L^p(\mathbb{R}^{d+1})} &= \int_{\mathbb{R}^{d+1}} |M_S^{e_{d+1}} f|^p = \int_{\mathbb{R}} \int_{\mathbb{R}^d} |M_S^{e_{d+1}} f(x,y)|^p dx\, dy \\
&= \int_{\mathbb{R}} \|M_S^{e_{d+1}} f(\cdot, y)\|^p_{L^p(\mathbb{R}^d)} dy \leq \int_{\mathbb{R}} C(d,p)^p \|f(\cdot, y)\|^p_{L^p(\mathbb{R}^d)} dy \\
&= C(d,p)^p \|f\|^p_{L^p(\mathbb{R}^{d+1})}.
\end{aligned}$$

Since $A_r^{\omega_0} f(x) = A_r^{e_{d+1}}(f \circ U_{\omega_0})(U_{\omega_0}^{-1}x)$, we see that

$$\|M_S^{\omega_0} f\|_{L^p(\mathbb{R}^{d+1})} \leq C(d,p)\|f \circ U_{\omega_0}\|_{L^p(\mathbb{R}^{d+1})} = C(d,p)\|f\|_{L^p(\mathbb{R}^{d+1})}. \tag{3.20}$$

The observation that lies at the heart of the "method of rotations" is that an average over an S^d is the same as an average over all averages of S^{d-1}s inside S^d. More rigorously:

Lemma 2. We have the following equality:

$$A_r f(x) = \int_{S^d} A_r^{\omega_0} f(x) d\sigma^d(\omega_0) \tag{3.21}$$

Proof. This observation follows from the general fact that given a compact lie group G acting transitively on a compact manifold M, there exists a unique G-invariant probability measure on M. We fix x and r, and have SO_{d+1} act transitively on the sphere $x + rS^d$. We define the two measures

$$\mu(E) = A_r\chi_E(x) = \int_{S^d} \chi_E(x - r\omega) d\sigma^d(\omega)$$
$$\nu(E) = \int_{S^d} A_r^{\omega_0}\chi_E(x) d\sigma^d(\omega_0) = \int_{S^d}\int_{S^{d-1}} \chi_E(x - rU_{\omega_0}\omega) d\sigma^{d-1}(\omega) d\sigma^d(\omega_0)$$

It follows immediately from the monotone convergence theorem that both μ and ν are in fact measures. It is clear that $\mu(x + rS^d) = \nu(x + rS^d) = 1$ and that both measures are SO_{d+1} invariant. Therefore, $\mu = \nu$, and integrating f against both of these measures gives the desired result. □

Taking supremums over r on both sides of (3.21) gives us that

$$M_S f(x) = \int_{S^d} A_r^{\omega_0} f(x) d\sigma^d(\omega_0) \le \int_{S^d} \sup_r A_r^{\omega_0} f(x) d\sigma^d(\omega_0) = \int_{S^d} M_S^{\omega_0} f(x) d\sigma^d(\omega_0).$$

Finally, using Minkowski's integral inequality and (3.20), we get that

$$\begin{aligned}\|M_S f(x)\|_{L^p(\mathbb{R}^{d+1})} &\le \left\|\int_{S^d} M_S^{\omega_0} f(x) d\sigma^d(\omega_0)\right\|_{L^p} \le \int_{S^d} \|M_S^{\omega} f\|_{L^p} d\sigma^d(\omega_0)\\ &\le \int_{S^d} C(d,p)\|f\|_{L^p(\mathbb{R}^{d+1})} d\sigma(\omega_0) = C(d,p)\|f\|_{L^p(\mathbb{R}^{d+1})}.\end{aligned}$$

This tells us that $C(d,p) = C(d+1,p)$, proving Theorem 5, and by the discussion in the beginning of this subsection, proves that the L^p norms of the Euclidean maximal function M can be taken independent of dimension for $p \in (1,\infty]$.

4 L^2 Bounds of Maximal Functions for Convex Bodies

Once Stein and Strömberg settled the independence of dimension for the maximal function over the ball in $\mathbb{R}^d$ with respect to the ℓ^2 norm. The natural follow up question is does this independence of dimension remain if we replace the ℓ^2 ball with unit balls with respect to other ℓ^p norms, for $p \in [1,\infty]$? More generally, if we choose an *arbitrary convex body* B in $\mathbb{R}^d$ of volume 1, fix $p \in (1,\infty]$ and consider the maximal function

$$Mf = M_B f = \sup_{r>0} \frac{1}{\mathrm{Vol}_d rB} \int_{rB} |f(x-y)| dy$$

where $rB = r \cdot B$, is it true that the L^p operator norm of M_B is independent of *both the dimension* n *and the convex body* B? Bourgain and Carbery showed that the answer is yes, but were only able to show this for $p > \frac{3}{2}$. In the following two sections, we will explain the proof of this claim, and see where the obstruction at $\frac{3}{2}$ arises. As before, the proof will boil down to obtaining an $L^2 \to L^2$ bound for M_B (and then interpolating with the trivial $L^\infty \to L^\infty$ bound), and interpolating this against an $L^p \to L^p$ for M_B. Throughout this section, we will derive an L^2 bound of M_B independent of dimension and the convex body B following Bourgain's paper [Bou86a].

We first work to set up the general framework around the maximal function M_B. For a general L^1 function $K : \mathbb{R}^d \to \mathbb{R}$, we can write

$$(f * K_{(t)})(x) = (f^\vee K_{[t]}^\vee)^\wedge = \int_{\mathbb{R}^d} \widehat{f}(-\xi)\widehat{K}_{[t]}(-\xi) e^{-2\pi i x\cdot\xi} d\xi = \int_{\mathbb{R}^d} \widehat{f}(\xi)\widehat{K}(t\xi) e^{2\pi i x\cdot\xi} d\xi.$$

Now, fixing a convex body $B \subset \mathbb{R}^d$ of volume 1, we write, by change of variables,

$$(\mathrm{Vol}_d tB)^{-1} \int_{tB} |f(x-y)| dy = (\mathrm{Vol}_d B)^{-1} \int_{tB} t^{-d} |f(x-y)| dy = f * (\chi_B)_{(t)}.$$

In this way, we may write the maximal operator M_B as the supremum of convolution operators

$$M_B f = \sup_{t>0} f * (\chi_B)_{(t)}.$$

Our goal is to bound the L^2 operator norm of the family of operators above independent of B and the dimension of the space it lives in. Just as in the proof of the independence of dimension on the spherical maximal function, we will use Littlewood-Paley theory to analyze this operator. Similarly to Stein's proof previously, we will obtain a bound on this operator norm as a function of the quantities

$$\alpha_j = \sup_{2^j \le t \le 2^{j+2}} |\widehat{K}(\xi)| \quad \text{and} \quad \beta_j = \sup_{2^j \le t \le 2^{j+2}} |\langle \nabla \widehat{K}(\xi), \xi \rangle|. \tag{4.1}$$

In the case of the spherical maximal inequality, we had decay of the symbol of the maximal operator due to Bessel function estimates, which utilize the geometry of the sphere. Here, we'll need to make use of the convexity of B in order to argue that the α_j, β_j are sufficiently small.

4.1 A general Littlewood-Paley bound

In this subsection, we will apply the Littlewood-Paley theory techniques from Stein's Spherical Maximal Theorem to a more general setting, for our general use later.

Theorem 6. Consider $K \in L^1(\mathbb{R}^d)$ and define α_j and β_j as in (4.1) above. Then for any $f \in \mathcal{S}(\mathbb{R}^d)$, there exists some universal constant C such that

$$\|\sup_{t>0} |f * K_{(t)}|\|_{L^2} \le C\Gamma(K)\|f\|_{L^2},$$

where we define

$$\Gamma(K) = \sum_{j\in\mathbb{Z}} \alpha_j^{\frac{1}{2}} (\alpha_j^{\frac{1}{2}} + \beta_j^{\frac{1}{2}}).$$

Proof. While the calculations for this proof become a bit messy, the ideas follow the same general Littlewood-Paley principles, but with decomposing the convolution kernel K rather than the input function f.

- We decompose K with a Littlewood-Paley decomposition into $\sum_{j\in\mathbb{Z}} k_j$
- We break up the supremum over t into the supremum over dyadic intervals
- We discretize the supremum $\sup_{2^v \le t \le 2^{v+1}} f * (k_j)_{(t)}$ over points spaced out by distances where $(k_j)_{(t)}$ is controlled

We create our Littlewood-Paley decomposition of K in the following way: let $\{\eta_j\}_{j=1}^{\infty}$ be a partition of unity of $\mathbb{R}_{>0}$, such that η_j is supported in $[2^j, 2^{j+2}]$, $0 \le \eta_j \le 1$, and $|\eta_j'| \le C2^{-j}$. With this, we define k_j to be the Fourier localizations of K onto an annulus of inner radius 2^j and outer radius 2^{j+2}, i.e.

$$\widehat{k_j}(\xi) = \eta_j(|\xi|)\widehat{K}(\xi).$$

Now by the triangle inequality,

$$\|\sup_{t>0} |f * K_{(t)}|\|_{L^2} \le \sum_j \|\sup_{t>0} |f * (k_j)_{(t)}|\|_{L^2}.$$

We fix a j, and split our supremum into dyadic intervals by square summing the supremum in v below.

$$\|\sup_{t>0} |f * (k_j)_{(t)}|\|_{L^2} = \|\sup_{v\in\mathbb{Z}} \sup_{2^v \le t \le 2^{v+1}} |f * (k_j)_{(t)}|\|_{L^2}$$

$$\le \left\| \left[\sum_{v\in\mathbb{Z}} \sup_{2^v \le t \le 2^{v+1}} |f * (k_j)_{(t)}|^2 \right]^{1/2} \right\|_{L^2} = \left[\sum_{v\in\mathbb{Z}} \| \sup_{2^v \le t \le 2^{v+1}} |f * (k_j)_{(t)}| \|_{L^2}^2 \right]^{1/2}. \tag{4.2}$$

We are left to analyze the behavior of

$$\|\sup_{2^v\le t\le 2^{v+1}}|f*(k_j)_{(t)}|\|_{L^2}. \tag{4.3}$$

We first express $f*(k_j)_{(t)}$ explicitly for $2^v\le t\le 2^{v+1}$:

$$\left|f*(k_j)_{(t)}(x)\right| = \left|\int_{\mathbb{R}^d}\widehat{f}(\xi)\widehat{k}_j(t\xi)e^{2\pi i x\cdot\xi}d\xi\right|$$

Since $\widehat{k}_j(\xi)$ is supported in $[2^j,2^{j+2}]$, and $2^v\le t\le 2^{v+1}$, we can replace $\widehat{f}(\xi)$ with $\widehat{f}_{j-v}(\xi)$, where we define for $m\in\mathbb{Z}$,

$$\widehat{f}_m(\xi)=\widehat{f}(\xi)\chi_{|\xi|\in[2^{m-1},2^{m+2}]}.$$

Now we are tasked with understanding the behavior of

$$\sup_{2^v\le t\le 2^{v+1}}\left|\int_{\mathbb{R}^d}\widehat{f}_{j-v}(\xi)\widehat{k}_j(t\xi)e^{2\pi i x\cdot\xi}d\xi\right|.$$

As discussed before, we discretize this sum, but we will leave the spacing in our net ambiguous and choose it at the end. We note that the differing treatments that we did for low and high frequencies in Stein's theorem will be taken care of through this spacing (which we can do since we're only working in the L^2 case). We fix an integer $A_j\ge 1$ and a v, and we consider a $(2^vA_j^{-1})$-net $\{t_\tau\}_{\tau\le A_j}$ of $[2^v,2^{v+1}]$. Again, this net is just a set of A_j points $\{t_1,t_2,\dots,t_{A_j}\}$, where t_τ and $t_{\tau+1}$ are spaced out on the order of $2^vA_j^{-1}$ from each other.

We recall the fundamental theorem of calculus identity (3.7). Plugging in $\left|\int_{\mathbb{R}^d}\widehat{f}_{j-v}(\xi)\widehat{k}_j(t\xi)e^{2\pi i x\cdot\xi}d\xi\right|$ for ϕ and $t_\tau,t_{\tau+1}$ for s_1,s_2 in the identity, we get that

$$\begin{aligned}&\sup_{t_\tau\le t\le t_{\tau+1}}\left|\int_{\mathbb{R}^d}\widehat{f}_{j-v}(\xi)\widehat{k}_j(t\xi)e^{2\pi i x\cdot\xi}d\xi\right|\\ &\le\left|\int_{\mathbb{R}^d}\widehat{f}_{j-v}(\xi)\widehat{k}_j(t_\tau\xi)e^{2\pi i x\cdot\xi}d\xi\right|+\int_{t_\tau}^{t_{\tau+1}}\left|\frac{d}{ds}\int_{\mathbb{R}^d}\widehat{f}_{j-v}(\xi)\widehat{k}_j(s\xi)e^{2\pi i x\cdot\xi}d\xi\right|ds.\end{aligned}$$

We note that the derivative of $\widehat{k}_j(s\xi)e^{2\pi i x\cdot\xi}$ with respect to s is $\langle\nabla\widehat{k}_j(s\xi),\xi\rangle$, which is L^1. By the mean value theorm and the Lebesgue dominated convergence theorem, we can pull the derivative into the integral. This gives us that

$$\begin{aligned}&\sup_{t_\tau\le t\le t_{\tau+1}}\left|\int_{\mathbb{R}^d}\widehat{f}_{j-v}(\xi)\widehat{k}_j(t\xi)e^{2\pi i x\cdot\xi}d\xi\right|\\ &\lesssim\left|\int_{\mathbb{R}^d}\widehat{f}_{j-v}(\xi)\widehat{k}_j(t_\tau\xi)e^{2\pi i x\cdot\xi}d\xi\right|+\left[\int_{t_\tau}^{t_{\tau+1}}\left|\int_{\mathbb{R}^d}\widehat{f}_{j-v}(\xi)\langle\nabla\widehat{k}_j(s\xi),\xi\rangle e^{2\pi i x\cdot\xi}d\xi\right|ds\right].\end{aligned} \tag{4.4}$$

Now we break up a supremum over $[2^v,2^{v+1}]$ as

$$\sup_{2^v\le t\le 2^{v+1}}\left|\int_{\mathbb{R}^d}\widehat{f}_{j-v}(\xi)\widehat{k}_j(t\xi)e^{2\pi i x\cdot\xi}d\xi\right|=\sup_\tau\sup_{t_\tau\le t\le t_{\tau+1}}\left|\int_{\mathbb{R}^d}\widehat{f}_{j-v}(\xi)\widehat{k}_j(t\xi)e^{2\pi i x\cdot\xi}d\xi\right|.$$

Plugging (4.4) into the above expression, dominating the supremum over τ with an ℓ^2 norm over τ, and putting everything past (4.3) together, we get that

$$\begin{aligned}\left\|\sup_{2^v\le t\le 2^{v+1}}|f*(k_j)_{(t)}|\right\|_{L^2}&=\left\|\sup_\tau\sup_{t_\tau\le t\le t_{\tau+1}}\left|\int_{\mathbb{R}^d}\widehat{f}_{j-v}(\xi)\widehat{k}_j(t\xi)e^{2\pi i x\cdot\xi}d\xi\right|\right\|_{L^2}\\ &\le\left(\sum_\tau\left\|\sup_{t_\tau\le t\le t_{\tau+1}}\left|\int_{\mathbb{R}^d}\widehat{f}_{j-v}(\xi)\widehat{k}_j(t\xi)e^{2\pi i x\cdot\xi}d\xi\right|\right\|^2\right)^{\frac12}\\ \le\left(\sum_\tau\left[\left\|\int_{\mathbb{R}^d}\widehat{f}_{j-v}(\xi)\widehat{k}_j(t_\tau\xi)e^{2\pi i x\cdot\xi}d\xi\right\|_{L^2}^2\right.\right.&\left.\left.+\left\|\int_{t_\tau}^{t_{\tau+1}}\left|\int_{\mathbb{R}^d}\widehat{f}_{j-v}(\xi)\langle\nabla\widehat{k}_j(s\xi),\xi\rangle e^{2\pi i x\cdot\xi}d\xi\right|ds\right\|_{L^2}^2\right]\right)^{\frac12}.\end{aligned} \tag{4.5}$$

Using Parseval, we estimate

$$\left\| \int_{\mathbb{R}^d} \widehat{f}_{j-v}(\xi)\widehat{k}_j(t_\tau \xi) e^{2\pi i x\cdot\xi} d\xi \right\|_{L^2} = \|f_{j-v} * (k_j)_{(t_\tau)}\|_{L^2} = \|\widehat{f}_{j-v}(\xi)\widehat{k}_j(t_\tau\xi)\|_{L^2} \le \|\widehat{k}_j\|_\infty \|f_{j-v}\|_{L^2}. \quad (4.6)$$

And using the same Parseval trick with Minkowski's integral inequality,

$$\left\| \int_{t_\tau}^{t_{\tau+1}} \left| \int_{\mathbb{R}^d} \widehat{f}_{j-v}(\xi)\langle \nabla \widehat{k}_j(s\xi), \xi\rangle e^{2\pi i x\cdot\xi} d\xi \right| ds \right\| = \int_{t_\tau}^{t_{\tau+1}} \left\| \int_{\mathbb{R}^d} \widehat{f}_{j-v}(\xi)\langle \nabla \widehat{k}_j(s\xi), \xi\rangle e^{2\pi i x\cdot\xi} d\xi \right\| ds$$
$$\le \int_{t_\tau}^{t_{\tau+1}} \|\langle \nabla \widehat{k}_j(s\xi), \xi\rangle\|_\infty \|f_{j-v}\|_{L^2} ds \le 2^v A_j^{-1} \|f_{j-v}\|_{L^2} \sup_{t_\tau \le t \le t_{\tau+1}} \|\langle \nabla \widehat{k}_j(t\xi), \xi\rangle\|_\infty. \quad (4.7)$$

Plugging (4.6) and (4.7) into (4.5) finally gives us that

$$\|\sup_{2^v \le t \le 2^{v+1}} |f * (k_j)_t|\|_{L^2} \le A_j^{\frac{1}{2}} (\|\widehat{k}_j\|_\infty \|f_{j-v}\|_{L^2} + 2^v A_j^{-1} \|f_{j-v}\|_{L^2} \sup_{2^v \le t \le 2^{v+1}} \|\langle \nabla \widehat{k}_j(t\xi), \xi\rangle\|_\infty)$$
$$\le A_j^{\frac{1}{2}} \|f_{j-v}\|_{L^2} (\|\widehat{k}_j\|_\infty + A_j^{-1} \|\langle \nabla \widehat{k}_j(\xi), \xi\rangle\|_\infty),$$

where the last step follows by bringing the 2^v into the inner product. We plug this result into (4.2) to get that

$$\|\sup_{t>0} |f * (k_j)_{(t)}|\|_{L^2} \le \|f\|_{L^2} (A_j^{\frac{1}{2}} \|\widehat{k}_j\|_\infty + A_j^{-\frac{1}{2}} \|\langle \nabla \widehat{k}_j(\xi), \xi\rangle\|_\infty).$$

The chain rule tells us that $\langle \nabla \widehat{k}_j(\xi), \xi\rangle = \langle \nabla \widehat{K}_j(\xi), \xi\rangle \eta_j(|\xi|) + \langle \widehat{K}_j(\xi), \xi\rangle \eta_j'(|\xi|)\frac{\xi}{|\xi|} \le C(\alpha_j + \beta_j)$, since we created η_j to control its derivative. If we pick $A_j = (\alpha_j + \beta_j)\alpha_j^{-1}$, we conclude that

$$\|\sup_{t>0} |f * K_{(t)}|\|_{L^2} \le C \left(\sum_{j\in\mathbb{Z}} \alpha_j^{\frac{1}{2}} (\alpha_j + \beta_j)^{\frac{1}{2}} \right) \|f\|_{L^2}.$$

□

4.2 Making use of convexity

In order to use Theorem 6 toward proving an L^2 bound on M_B, we need to argue that the α_j, β_j corresponding to $\widehat{\chi}_B$ (the convolution kernel associated to the maximal function M_B) are sufficiently small. As mentioned before, in Stein's Spherical Maximal Theorem, this quantity was bounded using the geometry of the sphere. In our current case, we will see that a strong understanding of $\widehat{\chi}_B$ will follow from an understanding of the volumes of the cross sections of B.

The key idea behind why we want to this is comes from the following observation: Our end goal is to understand the Fourier transform of χ_B, where B is a symmetric convex body. Then we are trying to control the integral

$$\widehat{\chi}_B(\xi) = \int_B e^{-2\pi i x\cdot\xi} dx.$$

Let's fix a $\xi \in \mathbb{R}^d$, and use Fubini to turn this integral into a double integral, integrating in both the ξ direction and in the hyperplane perpendicular to ξ. We note that frequencies oscillating in the ξ direction are going to be *constant* on hyperplanes perpendicular to ξ, since the inner product in the exponential is zero. Therefore, we can reduce the Fourier transform of χ_B to the one-dimensional integral dependent on the $d-1$-dimensional volume of cross sections of B perpendicular to ξ.

We give a brief overview of the convex geometry results that we'll use before moving on to their applications. Again, we assume that we are working with a symmetric convex body $B \in \mathbb{R}^d$ with $\mathrm{Vol}_d(B) = 1$. Then for ξ on the (ℓ^2) unit ball of $\mathbb{R}^d$, we define the function $\varphi_\xi : \mathbb{R} \to \mathbb{R}$ as

$$\varphi_\xi(u) = \mathrm{Vol}_{n-1}\{x \in B : \langle x, \xi\rangle = u\}$$

We note that the hyperplane $H_{\xi,u} = \{x \in \mathbb{R}^d : \langle x, \xi\rangle = u\}$ is a translate of the $d-1$ dimensional subspace orthogonal to ξ by the vector $u\xi$, so $\mathrm{Vol}_{n-1}\{x \in B : \langle x, \xi\rangle = u\}$ is the cross sectional volume of B sliced by the hyperplane $H_{\xi,u}$.

We quickly prove some simple properties of φ_ξ:

Lemma 3. For $B \subset \mathbb{R}^d$ a symmetric convex body, the function φ_ξ is decreasing on $[0, \infty)$, and $(\varphi_\xi)^{\frac{1}{d-1}}$ is concave outside of the region where $\varphi_\xi = 0$.

Proof. Concavity follows from the standard Brunn-Minkowski inequality: take $a, b \in \mathbb{R}$, and take any $0 \le p \le 1$. Then since $pH_{\xi,a} + (1-p)H_{\xi,b} \subset H_{\xi,pa+(1-p)b}$ by convexity, we have that

$$(\varphi_\xi(pa+(1-p)b))^{\frac{1}{d-1}} = \mathrm{Vol}_{n-1}(H_{\xi,pa+(1-p)b})^{\frac{1}{d-1}} \ge \mathrm{Vol}_{n-1}(pH_{\xi,a} + (1-p)H_{\xi,b})^{\frac{1}{d-1}}$$
$$\ge \mathrm{Vol}_{n-1}(pH_{\xi,a})^{\frac{1}{d-1}} + \mathrm{Vol}_{n-1}((1-p)H_{\xi,b})^{\frac{1}{d-1}} \ge p(\varphi_\xi(a))^{\frac{1}{d-1}} + (1-p)(\varphi_\xi(b))^{\frac{1}{d-1}}.$$

Since φ_ξ is an even function, we conclude that φ_ξ is decreasing on $[0, \infty)$. □

As discussed before, we want some uniform bound on the volumes of the cross sections of B. We would hope for a statement that says cross sections of B are "the same" up to some universal constant, i.e. there are constants $L(B)$ (depending on B) and C (independent of B) such that for all $\xi \in S^{d-1}$,

$$\frac{1}{C} \le L(B) \cdot \mathrm{Vol}_{d-1}(x \in v(B) : \langle x, \xi \rangle = 0) \le C$$

However, this is too strong to hope for. For instance, take a family of very long, skinny cylinders B_t in $\mathbb{R}^d$, of length t^{d-1} and cross sectional diameter $\frac{C}{t}$, where C is chosen so that each circular cross section has $d-1$ dimensional volume $\frac{1}{t^{d-1}}$. This is illustrated in the figure below.

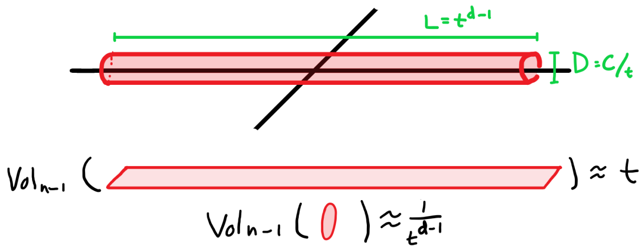

Figure 4: A family of cylinders whose largest to smallest cross section ratio goes to infinity

A horizontal slice of B_t gives us a cylinder one dimension lower, whose volume is on the order of t (since this is a cylinder of length t^{d-1} whose cross sections are $d-2$ dimensional spheres of radius t). Yet, a vertical slice of B_t gives us a $d-1$ dimensional sphere whose volume is on the order of $\frac{1}{t^{d-1}}$.

This result is true through if we allow ourselves to scale B by some linear transformation, that Bourgain credits to Milman.

Lemma 4. There is a $v \in SL(\mathbb{R}^d)$ and some constant L depending on B such that for all $\xi \in S^d$ and some universal constant C,

$$\frac{1}{CL} \le \mathrm{Vol}_{d-1}(x \in v(B) : \langle x, \xi \rangle = 0) \le \frac{C}{L}$$

In other words, given any convex body B, there exists some determinant 1 linear transformation v such that once we apply it to B, all of B's cross sections have essentially the same volume (up to a constant independent of B or the dimension)[2]. In the above Lemma, we can take $v \in SL(\mathbb{R}^d)$ to be such that $v(B)$ is in *isotropic position*, meaning that for all $\xi \in \mathbb{R}^d$,

$$\int_{v(B)} |\langle x, \xi \rangle|^2 dx = L^2 \|\xi\|_2^2. \tag{4.8}$$

[2]In fact, Lemma 4 actually holds without the constant L; that is, there exists a universal constant C such that for any convex body $B \subset \mathbb{R}^d$ of volume 1, there exists a linear transformation $v \in SL_n(\mathbb{R})$ such that all cross sections of $v(B)$ are bounded below by $\frac{1}{C}$ and above by C, where C is *independent of dimension and the convex body*! This statement, known as the Bourgain Slicing Conjecture, was proven by Klartag and Lehec in 2022 (see [KL22])

Such a v can be found as $\int_B |\langle x, \xi\rangle|$ is a positive definite quadratic form, hence is equivalent under $SL(\mathbb{R}^d)$ to a constant multiple the identity quadratic form $\|\xi\|^2$. The unique L in (4.8) is called the *isotropy constant* of B.

We note that by a simple change of variables, and the fact that $\mathrm{Vol}_d v(B) = \mathrm{Vol}_d B$ since $v \in SL(\mathbb{R}^d)$,

$$M_{v(B)}f(x) = \sup_{t>0}(\mathrm{Vol}_d v(B))^{-1} \int_{tv(B)} t^{-d}|f(x-y)|dy$$
$$= \sup_{t>0}(\mathrm{Vol}_d B)^{-1} \int_B t^{-d}|(f\circ v)(v^{-1}x - y)|dy = M_B(f\circ v)(v^{-1}x).$$

From here it is immediate that $\|M_B f\|_{L^2} = \|M_{v(b)}f\|_{L^2}$. Therefore, to show that M_B is bounded independent of dimension, we may assume that B is in isotropic position. We will make use of this in the next subsection.

4.3 The Poisson Kernel Trick

We apply the general Littlewood-Paley bounds from Theorem 6 to M_B, the convolution operator with kernel χ_B. To show that M_B is bounded independent of dimension and B, we need to show that for

$$\alpha_j = \sup_{2^j \le t \le 2^{j+2}} |\widehat{\chi}_B(\xi)| \qquad \beta_j = \sup_{2^j \le t \le 2^{j+2}} |\langle \nabla\widehat{\chi}_B(\xi), \xi\rangle|,$$

the sum

$$\sum_{j\in\mathbb{Z}} \alpha_j^{\frac{1}{2}}(\alpha_j^{\frac{1}{2}} + \beta_j^{\frac{1}{2}})$$

is bounded by a constant independent of B or the dimension of $\mathbb{R}^d$. To do so, as discussed before we first bound $\widehat{\chi_B}(\xi)$ in terms of it's cross-sectional volumes, where we may crucially assume that B is in isotropic position (by applying $v \in SL(\mathbb{R}^d)$) and thus satisfies the conclusion of Lemma 4. For any $\xi \in \mathbb{R}^d - \{0\}$, let ν denote the unit normal in the direction of ξ, and let $|\xi|$ denote the magnitude (ℓ^2 norm) of ξ. By Fubini and change of coordinates, splitting an integral over $\mathbb{R}^d$ into an integral over hyperplanes perpendicular to ν, we get that

$$\widehat{\chi}_B(\xi) = \int_B e^{-2\pi i x\cdot\xi}dx = \int_{\mathbb{R}} \varphi_\nu(u)e^{-2\pi i|\xi|u}du = \int_{\mathbb{R}} \varphi_\nu(u)e^{2\pi i|\xi|u}du,$$

where the last equality follows from the fact that B is symmetric about the origin, so φ is even. Since φ is an even function and sin is odd, we note that the imaginary part of this last integral is zero, leaving us with

$$\widehat{\chi}_B(\xi) = \int_{\mathbb{R}} \varphi_\nu(u)\cos(2\pi|\xi|u)du.$$

By integration by parts, and the fact that φ_ν is compactly supported,

$$|\widehat{\chi}_B(\xi)| = \left|\int_{\mathbb{R}} \varphi_\nu(u)\cos(2\pi|\xi|u)du\right| = \left|\int_{\mathbb{R}} \varphi'_\nu(u)\cos(2\pi|\xi|u)du\right| \le \int_{\mathbb{R}} |\varphi'_\nu(u)|du.$$

Since φ is even and positive we have that

$$\frac{C}{|\xi|}\int_{\mathbb{R}} |\varphi'_\nu(u)du| = \frac{2C}{|\xi|}\int_{\mathbb{R}_{\le 0}} \varphi'_\nu(u)du = \frac{2C}{|\xi|}\varphi(0) \sim \frac{1}{|\xi|}L^{-1}.$$

So in conclusion, we have that

$$|\widehat{\chi}_B(\xi)| \lesssim \frac{1}{|\xi|}L^{-1}. \tag{4.9}$$

However, this bound happens to not be strong enough, since we get that

$$\alpha_j \le \sup_{2^j \le t \le 2^{j+2}} \frac{1}{|\xi|}L^{-1} = \frac{1}{2^j L},$$

which as j goes to $-\infty$, causes α_j to blow up. The issue is more fundamental that this: for Theorem 6 to be useful out of the box, we need $\widehat{\chi_B}(\xi)$ to go to zero as $|\xi|$ goes to zero, but this isn't the case: we estimate $|1 - \widehat{\chi}_B(\xi)|$, by noting that

$$1 - \widehat{\chi}_B(\xi) = \int_B (1 - e^{2\pi i x\cdot\xi})dx = \int_{\mathbb{R}} \varphi_\nu(u)(1 - \cos(2\pi|\xi|u))du.$$

We have again by integration by parts that

$$|1 - \chi_B| \leq C \int_{\mathbb{R}} \varphi_\nu(u)|u||\xi| \leq C_2|\xi|L. \tag{4.10}$$

This tells us that $\widehat{\chi_B}(\xi)$ goes to 1 as $|\xi| \to 0$, and so our sum in α_j cannot converge as $j \to -\infty$. Bourgain's insight was to not bound $\widehat{\chi_B}$ directly with the estimates in Theorem 6, but to instead bound $(\chi_B - T)^\wedge$ for T an operator such that

1. The maximal operator $\sup_{t>0} |T_{(t)} * f|$ is L^2 bounded
2. The Fourier transform $\widehat{T}(\xi)$ goes to 1 as $|\xi| \to 0$

For our operator T, we choose the Poisson kernel P_L, with L the isotropy constant of B, defined on the Fourier side by

$$\widehat{P}_L(\xi) = e^{-2\pi L|\xi|}$$

It is clear that $\widehat{P}_L(\xi) \to 1$ as $|\xi| \to 0$. The fact that the maximal operator $\sup_{t>0} T_t$ is L^2 bounded follows from the semigroup maximal theorem below.

For a fixed $p \in [1, \infty]$, a *semigroup of operators* is a one-parameter family of L^p operators $\{T_t\}_{t\in\mathbb{R}_{\geq 0}}$ that is a semigroup, i.e.

- $T_{t_1} \circ T_{t_2} = T_{t_1+t_2}$
- $T_0 = \text{Id}$

We have the following maximal theorem for semigroups satisfying the following axioms (known as symmetric diffusion semigroups):

Theorem 7. (The General Semigroup Maximal Theorem in $\mathbb{R}^d$)
Consider a semigroup $\{T_t\}_{t\in\mathbb{R}_{>0}}$ of $L^p(\mathbb{R}^d)$ operators that satisfies the following axioms:

1. T_t is a contraction, i.e. $\|T_t f\|_p \leq \|f\|_p$
2. T_t is symmetric, i.e. is a self-adjoint operator on $L^2(\mathbb{R}^d)$
3. T_t is positive, i..e $T_t f \geq 0$ if $f \geq 0$
4. $\lim_{t\to 0} \|T_t f\|_{L^2} = \|f\|_{L^2}$
5. $T_t 1 = 1$

Then the maximal function $M_T(f) = \sup_{t>0} |T_t f(x)|$ is bounded as an operator from $L^p \to L^p$ by some constant A_p independent of the dimension $\mathbb{R}^d$.

Proof. See page 73 of [Ste70]. □

Clearly, the family $T_t = (P_L)_{(t)}$ for $t > 0$ is a semigroup satisfying all of the properties above. Using this, we apply the general Littlewood-Paley theory estimates from Theorem 6 to $K = \chi_B - P_L$, where $L = L(B)$ is the isotropy constant of B, *instead of applying it directly to* χ_B. If we're able to show that independent of dimension and choice of B,

$$\|\sup_{t>0} |f * K_{(t)}|\|_2 \leq C\|f\|_2,$$

it immediately follows from the triangle inequality and the maximal theorem for semigroups that M_B is bounded independent of B and the dimension d.

To show such a C exists, it remain to compute the α_i, β_i in Theorem 6 for K. We first obtain a bound on $|\langle \nabla \widehat{\chi}_B(\xi), \xi\rangle|$. Since the derivative $e^{2\pi i x\cdot\xi}$ is integrable on B, by Lebesgue Dominated Convergence and the mean value theorem we get that

$$\langle \nabla \widehat{\chi}_B(\xi), \xi\rangle = \langle \int_B \nabla_\xi e^{2\pi i x\cdot\xi} dx, \xi\rangle = \langle \int_B 2\pi i x e^{2\pi i x\cdot\xi} dx, \xi\rangle = C\int_B \langle x, \xi\rangle e^{2\pi i x\cdot\xi} dx.$$

Again by Fubini and change of coordinates, we get

$$\int_B \langle x, \xi\rangle e^{2\pi i x\cdot\xi} dx = \int_{\mathbb{R}} u|\xi|\varphi_\alpha(x) e^{2\pi i u|\xi|} du,$$

and integrating by parts twice gives us

$$|\langle \nabla \widehat{\chi}_B(\xi), \xi\rangle| \le C\int_{\mathbb{R}} |(u\varphi(u))'| du \le C_1 \left(\int_{\mathbb{R}} \varphi(u) du + \int_{\mathbb{R}} |u\varphi'(u)| du \right) \le 2C_1. \tag{4.11}$$

By (4.11), $|\langle \nabla \widehat{\chi}_B(\xi), \xi\rangle|$ is uniformly bounded by a constant. Since $|\langle \nabla \widehat{P}_L(\xi), \xi\rangle|$ is uniformly bounded by a constant as well, we have that β_j is uniformly bounded in j. To bound α_j, we work in the two regimes $2^j \le L^{-1}$ and $2^j \ge L^{-1}$.

- In the regime $2^j \le L^{-1}$,

$$\begin{aligned}\alpha_j = \sup_{2^j \le |\xi| \le 2^{j+2}} |(\widehat{\chi_B - P_L})(\xi)| &\le \sup_{2^j \le |\xi| \le 2^{j+2}} (|1 - \widehat{\chi}_B(\xi)| + |1 - \widehat{P}_L(\xi)|) \\ &\lesssim 2^j L + (1 - e^{-2\pi 2^{j+2} L}) \lesssim 2^j L\end{aligned}$$

- In the regime $2^j \ge L^{-1}$,

$$\alpha_j = \sup_{2^j \le |\xi| \le 2^{j+2}} (|\widehat{\chi}_B(\xi)| + |\widehat{P}_L(\xi)|) \lesssim 2^{-j} L^{-1} + e^{-2\pi L 2^j} \lesssim 2^{-j} L^{-1}$$

From here, it is clear that the sum $\sum_{j\in\mathbb{Z}} \alpha_j^{\frac{1}{2}} (\alpha_j^{\frac{1}{2}} + \beta_j^{\frac{1}{2}})$ converges to a value *independent of* L, and we conclude the proof that M_B is strong $(2,2)$ bounded independent of B and the dimension of $\mathbb{R}^d$.

5 L^p Bounds of Maximal Functions for Convex Bodies

From the obvious L^∞ bound on M_B independent of dimension and B, interpolation with the dimension-free L^2 bounds for M_B just proved give dimension-free L^p boundedness of M_B independent of B for $p \ge 2$. The same argument for the Euclidean maximal function shows that M_B is not L^1 for any convex body B, but it remains to understand what happens in the regime $p \in (1,2)$. Shortly after Bourgain proved the dimension-free L^2 bounds for M_B in the previous section, both Bourgain ([Bou86b]) and Carbery ([Car86]) independently extended this result to L^p for $p > \frac{3}{2}$.

Bourgain's approach to $p > \frac{3}{2}$ analyzes the *dyadic maximal operator* associated to $B \subset \mathbb{R}^d$:

$$M_{B,1} f(x) = \sup_{j\in\mathbb{Z}} \frac{1}{\mathrm{Vol}_d(B)} \int_B |f(x - 2^j y)| dy.$$

This object is more convenient to study than the standard maximal function, since we are taking a discrete supremum rather than an continuous one, although it is clear that L^p boundedness of $M_{B,1}$ is weaker than that of M_B. To analyze this object, Bourgain considers the vector-valued operator $f_j * (\chi_B)_{(2^j)}$, which takes in a sequence of functions $(f_j)_{j\in\mathbb{Z}}$ and outputs the sequence of functions $(f_j * (\chi_B)_{(2^j)})_{j\in\mathbb{Z}}$. By studying the $L^p(\ell^q)$ norms of this vector-valued operator, i.e the constants $A(p,q)$ such that

$$\left\| \left(\sum_j (f_j * (\chi_B)_{(2^j)})^q \right)^{\frac{1}{q}} \right\|_{L^p} \le A(p,q) \left\| \left(\sum_j (f_j)^q \right)^{\frac{1}{q}} \right\|_{L^p},$$

vector-valued interpolation and Fourier analysis leads to the result that $A(p,\infty) < \infty$ for all $p \in (1,\infty)$. This immediately implies the result that for all $p \in (1,\infty)$, there exists a C such that

$$\|M_{B,1}f(x)\|_{L^p} \lesssim_p C\|f\|_{L^p}.$$

Bourgain then presents a short lemma that allows one to relate the norms of $M_{B,1}$ with M_B, which is only strong enough to obtain L^p bounds of M_B independent of dimension and B for $p > \frac{3}{2}$. Interpolating this result with Bourgain's L^2 estimates in the previous section gives L^p boundedness of M_B independent of dimension and B for all $p > \frac{3}{2}$.

In this section, we choose to exposit Carbery's argument ([Car86]) in full rather than Bourgain's. This is primarily due to the fact that future work in the area of maximal functions build on the framework developed in Carbery's article, in particular Müller's independence of dimension result in 1990 for the maximal function associated to ℓ^q balls for $q \in [1,\infty)$, and Bourgain's analogous result in 2014 for the case $q = \infty$. Along with Carbery's original paper, we also draw from material in a survey by Deleaval, Guedon, and Maurey, particularly Chapters 6 and 7 ([LM18]).

In Carbery's approach, an interpolation approach to maximal operators between L^p and L^2 is introduced, that rather than requiring equal strength in both bounds on M_B, requires more strength on the L^2 side and less strength on the L^p side. The required estimates needed to apply this interpolation will arise as consequences of boundedness of *fractional derivatives*, which we will discuss later in this section.

5.1 Carbery's Maximal Function Interpolation

Given a family of linear operators T_{jv} indexed for $j \in \mathbb{Z}$ and v any indexing set, and R_k for $k \in \mathbb{Z}$ some Littlewood-Paley decomposition of $\mathbb{R}_{>0}$, we consider the corresponding maximal operator $T_* = \sup_j \sup_v |T_{jv}f|$. The maximal function we are looking to show L^p boundedness for, $\sup_{t>0}(\chi_B - P_L)_{(t)} * f$ with B a symmetric convex body of volume 1 and L it's isotropy constant, fits into this framework by setting $T_{jv}f = (\chi_B - P_L)_{(2^j v)} * f$ for $j \in \mathbb{Z}$ and $v \in [1,2]$[3].

We say this family is *strongly bounded* on L^p with respect to the Littlewood-Paley decomposition R_k if

$$\|\sup_j \sup_v |T_{jv}R_{j+k}f|\|_{L^p} \le a_k\|f\|_{L^p}, \text{where } \sum_{k\in\mathbb{Z}} a_k^t < \infty \text{ for all } t \in (0,1]$$

Strong boundedness implies boundedness for the maximal operator T_* by the triangle inequality. In our setting, $T_{jv}f = (\chi_B - P_L)_{(2^j v)} * f$, strong boundedness is asking that as the scale of frequencies f is localized at differs more and more from the spatial scale T_{jv} is averaging on, $\|T_{jv}f\|_{L^p}$ should decay sufficiently fast. This is another reflection of the almost-orthogonality principal of Littlewood-Paley theory. We note that both the techniques of the spherical maximal theorem and Bourgain's dimension-free L^2 bounds have not produced strong L^2 boundedness for their corresponding families of maximal operators.

We say that the family of operators T_{jv} is *weakly bounded* on L^p with respect to the Littlewood-Paley decomposition R_k if

$$\sup_k \|\sup_j \sup_v |T_{jv}R_{j+k}f|\|_{L^p} \le C\|f\|_{L^p}$$

Again, in our setting, $T_{jv}f = (\chi_B - P_L)_{(2^j v)} * f$, being weakly bounded is asking that no matter the difference between the scales T_{jv} is averaging on and the frequencies of f are localized at, $T_{jv}f$ is never "too large." It is easy to check that boundedness of T_* implies weak boundedness of the family T_{jv}.

We have the following interpolation result between strong and weak bounds of a family of linear operators:

> **Lemma 5.** For $q_0 < q < q_1$, if the family of linear operators T_{jv} is strongly bounded on L^{q_0} and weakly bounded on L^{q_1}, then T_{jv} is bounded on L^q for all $q \in (q_0, q_1)$, with constant independent of dimension.

[3]We keep the Poisson term in our convolution kernel to aid in our estimates later

Proof. We fix a $q \in (q_0, q_1)$. For each k, we use Marcinkiewicz interpolation on each of the sublinear operators $\sup_j \sup_v |T_{jv} R_{j+k}|$ to get that for $\alpha = \frac{\frac{1}{q_0} - \frac{1}{q}}{\frac{1}{q_0} - \frac{1}{q_1}}$ and a_k independent of dimension,

$$\|\sup_j \sup_v |T_{jv} R_{j+k} f|\|_{L^q} \lesssim_q a_k^\alpha \|f\|_{L^q}$$

Since $\alpha \in (0,1)$, strong boundedness gives us that $\sum_{k\in\mathbb{Z}} a_k^\alpha < \infty$. Then by the triangle inequality, we conclude that

$$\|\sup_j \sup_v |T_{jv}|\|_{q\to q} \leq \sum_{k\in\mathbb{Z}} \|\sup_j \sup_v |T_{jv} R_{j+k}|\|_{q\to q} < \infty$$

□

Carbery's main proposition in his paper is a collection of weaker conditions that allows one to do an interpolation argument as above. To prove T_* is L^q bounded for $q \in (q_0, q_1)$, we will need to show that T_{jv} is a strongly L^{q_1} bounded family of operators, but we can get away with less than weak boundedness for the control we need in L^{q_0}. The statement of the Theorem below is slightly less general than Carbery's statement in [Car86], but it is all that will be used in this exposition.

Theorem 8. (Carbery's Interpolation Theorem) Suppose $q \in (1,2)$, T_{jv} is a family of linear operators on measurable functions from $\mathbb{R}^d$ to $\mathbb{R}$, and R_k is a Littlewood-Paley decomposition. If we have that

1. T_* is *essentially positive*, i.e. $T_{jv} = U_{jv} - S_{jv}$, with U_{jv}, S_{jv} a family of positive operators, and for all $r \in (q, 2]$, S_* is L^r bounded by a constant K_r
2. For all $r \in (q, 2]$, we have that
$$\|(\sum_{k\in\mathbb{Z}} |R_k f|^2)^{\frac{1}{2}}\|_{L^r} \leq C'_r \|f\|_{L^r}$$
3. T_* is strongly bounded on L^2 by some constant K
4. For all $r \in (q, 2]$, we have that
$$\sup_j \|\sup_v |T_{jv} f|\|_{L^r} \leq C_r \|f\|_{L^r}$$

Then in fact T_* is L^p bounded for all $p \in (q, 2]$, with operator norm bounded above as a function of K_r, K, C_r, C'_r

Proof. We fix a $p \in (q, 2)$. To show that T_* is L^p bounded, we first consider a truncated version of this operator, denoted as $T_*^{(N)}$, where we take the supremum of T_{jv} over all v but only over $j \in \mathbb{Z} \cap [-N, N]$. For a fixed N we get that

$$\|T_*^{(N)}\|_p = \|\max_{j\in\mathbb{Z}\cap[-N,N]} \sup_v T_{jv}\|_p \leq \left\| \left(\sum_{j=-N}^{N} (\sup_v T_{jv})^p \right)^{\frac{1}{p}} \right\|_{L^p} \leq (2N)^{\frac{1}{p}} \|\sup_v T_{jv}\|_p \leq A(N)\|f\|_p.$$

We will show that this constant $A(N)$ above is in fact independent of N, and thus we can take a limit in N to conclude that T_* is L^p bounded.

We fix r_0, r_1 such that $q < r_0 < r_1 < p < 2$. We also define a vector-valued version of the operator $\sup_v |T_{jv} f|$, which takes in a sequence of functions $(g_j)_{j\in\mathbb{Z}\cap[-N,N]}$ and outputs the sequence $(\sup_v |T_{jv} g_j|)_{j\in\mathbb{Z}\cap[-N,N]}$. We will study $L^s(\ell^t)$ estimates of this vector-valued operator. We have that when when $s = t = r_0$,

$$\left\|\left\|\sup_v |T_{jv}g_j|\right\|_{\ell^{r_0}}\right\|_{L^{r_0}} = \left\|\left(\sum_{j=-N}^{N} \sup_v |T_{jv}g_j|^{r_0}\right)^{\frac{1}{r_0}}\right\|_{L^{r_0}} = \left(\sum_{j=-N}^{N} \|\sup_v |T_{jv}g_j|\|_{L^{r_0}}^{r_0}\right)^{\frac{1}{r_0}}$$

$$\leq \left(\sum_{j=-N}^{N} (C_{r_0})^{r_0}\|g_j\|^{r_0}\right)^{\frac{1}{r_0}} = C_{r_0}\|\|g_j\|_{\ell^{r_0}}\|_{L^{r_0}}.$$

We now consider $s=\infty, t=r_1$. We make the following decomposition of T_{jv},

$$\sup_{j\in\mathbb{Z}\cap[-N,N]} \sup_v |T_{jv}g_j| \leq \sup_{j\in\mathbb{Z}\cap[-N,N]} \sup_v |S_{jv}g_j| + \sup_{j\in\mathbb{Z}\cap[-N,N]} \sup_v |U_{jv}g_j|$$

where we treat S and U as vector-valued. If we let $g = \sup_{j\in\mathbb{Z}\cap[-N,N]}|g_j|$ and consider a sublinear positive operator W, we note that $|Wg_j| \leq W|g_j| \leq Wg$. Since S is sublinear as a supremum of linear operators, we get that

$$\left\|\left\|\sup_v |S_{jv}g_j|\right\|_{\ell^\infty}\right\|_{L^{r_1}} = \|\sup_{j\in\mathbb{Z}\cap[-N,N]} \sup_v |S_{jv}g_j|\|_{L^{r_1}} \leq \|\sup_v |S_{jv}g|\|_{L^{r_1}} \leq K_{r_1}\|g\|_{L^{r_1}}.$$

Using the fact that U is sublinear as well, we get that

$$\left\|\left\|\sup_v |U_{jv}g_j|\right\|_{\ell^\infty}\right\|_{L^{r_1}} = \|\sup_{j\in\mathbb{Z}\cap[-N,N]} \sup_v |U_{jv}g_j|\|_{L^{r_1}} \leq \|\sup_v |U_{jv}g|\|_{L^{r_1}}$$

$$\leq \|\sup_v |S_{jv}g|\|_{L^{r_1}} + \|\sup_v |T_{jv}g|\|_{L^{r_1}} \leq (K_{r_1} + \|T_*^{(N)}\|_{L^{r_1}})\|g\|_{L^{r_1}}.$$

We conclude that

$$\left\|\left\|\sup_v |T_{jv}g_j|\right\|_{\ell^\infty}\right\|_{L^{r_1}} \| \sup_{j\in\mathbb{Z}\cap[-N,N]} \sup_v |S_{jv}g_j\|_{L^{r_1}} \leq (2K_{r_1} + \|T_*^{(N)}\|_{L^{r_1}})\|g\|_{L^{r_1}}.$$

We now apply the vector-valued Marcinkiewicz interpolation theorem (a corollary of the standard Marcinkiewicz interpolation, see for instance Exercise 4.5.3 in [Gra14]). We interpolate our $L^{r_0}(\ell^{r_0})$ and $L^{r_1}(\ell^\infty)$ bounds to get that $\sup_v |T_{jv}g_j|$ is $L^{r_2}(\ell^2)$ bounded for some $r_2 \in (r_0, r_1)$, with constant $K'(2K_{r_1} + \|T_*^{(N)}\|)^\alpha$ for K' some constant depending on r_1, r_0, and $\alpha < 1$.

Now we set g_j to R_{j+k}, to get that

$$\left\|\max_{j\in\mathbb{Z}\cap[-N,N]} \sup_v |T_jR_{j+k}f|\right\|_{L^r} \leq \left\|\sum_{j=-N}^{N}\left(\sup_v |T_jR_{j+k}f|^2\right)^{\frac{1}{2}}\right\|_{L^r}$$

$$\leq K'(2K_{r_1} + \|T_*^{(N)}\|)^\alpha \left\|\left(\sum_{j=-N}^{N} R_{j+k}f\right)^{\frac{1}{2}}\right\|_{L_r} \leq C_r' K'(2K_{r_1} + \|T_*^{(N)}\|)^\alpha \|f\|_{L^r}.$$

We therefore conclude that $T_*^{(N)}$ is weakly bounded as a family of L^{r_2} operators. Using Condition 3 of the theorem and Lemma 5 we get that $T_*^{(N)}$ is L^p bounded for our distinguished p fixed in the beginning of the proof, with constant

$$\|T_*^{(N)}\|_{L^p} \leq D_1(D_2 + D_3\|T_*^{(N)}\|_{L^p})^{\alpha'}$$

for $\alpha' < 1$, and D_1, D_2, D_3 all independent of N. Therefore, it is clear that we can take a bound for $\|T_*^{(N)}\|_{L^p}$ indepdendent of N.

Since for a fixed f, $T_*^{(N)}f$ is an increasing sequence of positive functions that pointwise converge to T_*f as $N \to \infty$, we get by the Monotone Convergence Theorem that $\lim_{n\to\infty}\|T_*^{(N)}f\|_{L^p} = \|T_*f\|_{L^p}$. Therefore, we get that T_* is L^p bounded, and we are done. □

For the rest of this section, for B a symmetric convex body of volume 1 we consider the family of operators

$$T_{jv}f = (\chi_B - P_L)_{(2^j v)} * f \tag{5.1}$$

for $j \in \mathbb{Z}$ and $v \in [1,2]$. The fact that M_B is L^p bounded independent of dimension and B for $p > \frac{3}{2}$ follows immediately if T_{jv} satisfies the four conditions needed for Carbery's interpolation theorem with constants independent of dimension. We note that both $(\chi_B)_{2^j v}$ and $(P_L)_{2^j v}$ are positive operators for all j, v, and by the semigroup maximal theorem, we see that Condition 1 is met independent of dimension. To meet Condition 2, we consider the Littlewood-Paley decomposition given by

$$R_j = P_{(2^{j+1})} - P_{(2^j)}, \tag{5.2}$$

where P is again the Poisson kernel given by $\widehat{P}(\xi) = e^{-2\pi|\xi|}_{L^2}$. Writing R_j as $\int_{2^j}^{2^{j+1}} \frac{d}{dt} P_{(t)} f(x) dt$ and applying Cauchy-Schwartz, we have that

$$|R_j|^2 \leq 2^j \int_{2^j}^{2^{j+1}} \left| \frac{d}{dt} P_{(t)} f(x) \right| dt \leq 2^j \int_{2^j}^{2^{j+1}} \left| \frac{d}{dt} P_{(t)} f(x) \right| dt$$

Therefore, we see that

$$\left(\sum_{j\in\mathbb{Z}} |R_j|^2 \right)^{1/2} \leq \left(\int_0^\infty t \left| \frac{d}{dt} P_{(t)} f(x) \right|^2 dt \right)^{1/2}$$

The function on the right is *Littlewood-Paley function* $g_1(f)$ associated to the Poisson semigroup. It is a classical result of Stein (see [Ste83]) that g_1 has L^p operator norm bounded in $p \in (1,2]$ and independent of dimension. Therefore, we conclude that Condition 2 holds independent of dimension.

What remains is to establish criteria to check when Conditions 3 and 4 of Carbery's interpolation theorem are satisfied. Both of these criteria will involve fractional differentiation and integration, so in the following section we develop the preliminaries that are necessary.

5.2 Fractional Derivative Techniques

Given a Schwartz function $h : \mathbb{R} \to \mathbb{R}$, we know there exists a Schwartz function $k : \mathbb{R} \to \mathbb{R}$ such that $\widehat{k} = h$. By properties of the Fourier transform, we can write $k^{(j)}(t)$, the jth derivative of k, as $((-2\pi i s)^j k(s))^\vee$. This allows us to express the jth derivative of h as follows:

$$h^{(j)}(t) = (-1)^j \int_{\mathbb{R}} (-2\pi i s)^j k(s) e^{-2\pi i s t} ds$$

With this, we define for every $\mathrm{Re}(z) > -1$ the fractional derivative operator D^z for $z \in \mathbb{C}$ as

$$(D^z h)(t) = \int_{\mathbb{R}} (2i\pi s)^z k(s) e^{-2\pi i s t} ds. \tag{5.3}$$

When $z = 1$, $D^z h(t)$ reduces to $-h'(t)$. In the same way, we define the fractional integration operator $I^\omega h(t)$ as a generalization of Cauchy's formula for repeated integration: for $\omega \in \mathbb{C}$, $\mathrm{Re}(\omega) > 0$, and $t > 0$, we define the fractional integration operator I^ω as

$$(I^\omega h)(t) = \frac{1}{\omega} \int_t^\infty (u-t)^{\omega-1} h(u) du. \tag{5.4}$$

When it is unclear what variable in a function we are fractional differentiating or integrating, we will add it in a subscript for clarity.

We would hope for fractional integration and differentiation to be inverse operators. In this way, we give a second definition of the fractional derivative operator. We analytically continue (5.4) by integrating by parts to get a formula for $I^\omega h(t)$ valid for $\mathrm{Re}(\omega) > -1$, giving us that

$$(I^\omega h)(t) = -\frac{1}{1+\omega} \int_t^\infty (u-t)^\omega h'(u) du.$$

For $\mathrm{Re}(z) \in (0,1)$ and $t > 0$, we denote temporarily $(\widetilde{D}^z h)(t)$ as $(I^{-z}h)(t)$, i.e.

$$(\widetilde{D}^z h)(t) = -\frac{1}{\Gamma(1-z)} \int_t^\infty (u-t)^{-z} h'(u) du. \tag{5.5}$$

It turns out that under sufficient regularity conditions of h, definitions (5.3) and (5.5) are equivalent. Throughout this section, we will often require the following regularity on h defined on $(0, \infty)$:

$$\begin{cases} h \text{ is Lipschitz on } (0,\infty) \\ |h(t)| \leq C_0(1+|t|)^{-1} \text{ for all } t > 0 \text{ and } C_0 \text{ independent of } t \\ |h'(t)| \leq C_1(1+|t|)^{-1} \text{ for almost all } t > 0 \text{ and } C_1 \text{ independent of } t \end{cases} \tag{5.6}$$

Claim 1. For $z \in \mathbb{C}, \mathrm{Re}(z) \in (0,1)$, h satisfying (5.6), and $t > 0$, $D^z h$ and $\widetilde{D}^z h$ agree. Equivalently, for $\widehat{k} = h$,

$$-\frac{1}{\Gamma(1-z)} \int_t^\infty (u-t)^{-z} h'(u) du = \int_{\mathbb{R}} (2i\pi s)^z k(s) e^{-2\pi i s t} ds$$

Having both these definitions will be useful, as we will often take k to be a convolution kernel where we have more information about it's Fourier transform h. Under the regularity stated previously, the following also holds:

Claim 2. For $\alpha \in (0,1)$ and h satisfying (5.6), we have that $(I^\alpha D^\alpha h)(t) = t$.

We also have the following simple bound that will be of use to us later:

Claim 3. Let $\alpha \in (0,1)$ and suppose $h : (0,\infty) \to \mathbb{R}$ satisfies the regularity conditions in (5.6). If h is decreasing and concave on $(0,\infty)$, then we have that

$$|(D^\alpha h)(t)| \lesssim h(t)^{1-\alpha} h'(t)^\alpha$$

For proofs of the preceding claims, see section 6.2 of [LM18].

Now that we've defined the fractional derivative for $h : \mathbb{R} \to \mathbb{R}$ where $h = \widehat{k}$, we now work to define the directional fractional derivative of the symbol of a Fourier multiplier on $L^p(\mathbb{R}^d)$. For $K \in L^1(\mathbb{R}^d)$, we define the convolution operator $K * f$, which by Young's inequality is L^p bounded for all $p \in [1,\infty)$. By writing $\xi = |\xi|\theta$, where $|\cdot|$ denotes the ℓ^2 norm, and letting $m(\xi) = \widehat{K}(\xi)$ we get by Fubini that

$$m(\xi) = \int_{\mathbb{R}^d} K(x) e^{-2\pi i x\cdot\xi} dx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{d-1}} K(y + s\theta) dy \right) e^{-2\pi i s u|\xi|} ds.$$

If we write φ_θ as $\int_{\mathbb{R}^{d-1}} K(y+s\theta)dy$ then we have for $\xi \neq 0$,

$$m(u\xi) = \int_{\mathbb{R}} \varphi_\theta(s) e^{-2\pi i s u|\xi|} ds = \int_{\mathbb{R}} \frac{1}{|\xi|} \varphi_\theta\left(\frac{v}{|\xi|}\right) e^{-2\pi i v u} dv.$$

Therefore, we see that the Fourier transform of $\frac{1}{|\xi|}\varphi_\theta\left(\frac{v}{|\xi|}\right)$ in u is $m(u\xi)$. With this, assuming that $|x|^\alpha K(x) \in L^1(\mathbb{R}^d)$ we get that

$$D_u^\alpha m(u\xi) = \int_{\mathbb{R}} (2\pi i v)^\alpha \frac{1}{|\xi|} \varphi_\theta\left(\frac{v}{|\xi|}\right) e^{-2\pi i u v} dv = \int_{\mathbb{R}^d} (2\pi i x\cdot\xi)^\alpha K(x) e^{-2\pi i u x\cdot\xi} d\xi.$$

In this way we define the directional fractional derivative operator as

$$(\xi\cdot\nabla)^\alpha m(\xi) := D_u^\alpha m(u\xi)\Big|_{u=1} = \int_{\mathbb{R}^d} (2\pi i x\cdot\xi)^\alpha K(x) e^{-2\pi i x\cdot\xi} dx. \tag{5.7}$$

When $0 < \alpha < 1$, we can apply Claim 1 to get the equivalent definition

$$(\xi\cdot\nabla)^\alpha m(\xi) = -\frac{1}{\Gamma(1-\alpha)} \int_1^\infty (u-1)^{-\alpha} \frac{d}{du} m(u\xi) du. \tag{5.8}$$

Setting $\alpha = 1$ and $\xi \neq 0$, we note that $(\xi \cdot \nabla) m(\xi)$ is just $-\xi \cdot \nabla m(\xi)$, the directional derivative of m with respect to ξ.

Lastly, given a smooth function h with compact support in $(0, \infty)$, we define for $\alpha \in (0,1)$ the norm

$$\|h\|_{L^2_\alpha} := \left(\int_0^\infty \left| t^{\alpha+1} D^\alpha \left(\frac{h(t)}{t} \right) \right|^2 \frac{dt}{t} \right)^{1/2}.$$

We note that this norm is invariant under dilation, which can be seen as follows: by using the Fourier definition (5.3) of the fractional derivative, we have that

$$t^{\alpha+1} D_t^\alpha \left(\frac{h(\lambda t)}{t} \right) = (\lambda t)^{\alpha+1} D_v^\alpha \left(\frac{h(\lambda v)}{\lambda v} \right) \Big|_{v=\lambda t}. \tag{5.9}$$

Substituting this into the definition above gives us

$$\|h_{[\lambda]}\|_{L^2_\alpha} := \left(\int_0^\infty \left| (\lambda t)^{\alpha+1} D_v^\alpha \left(\frac{h(\lambda v)}{\lambda v} \right) \Big|_{v=\lambda t} \right|^2 \frac{dt}{t} \right)^{1/2} = \left(\int_0^\infty \left| t^{\alpha+1} D^\alpha \left(\frac{h(t)}{t} \right) \right|^2 \frac{dt}{t} \right)^{1/2} = \|h\|_{L^2_\alpha},$$

where the middle equality follows from the change of variables $u = \lambda t$ after expanding D_v^α with the Fourier definition (5.3) of the fractional derivative.

In the following two lemmas, we connect fractional derivatives with L^2 and L^p bounds on maximal operators. We will use these lemmas to check when Conditions 3 and 4 of Carbery's Interpolation Theorem hold.

Lemma 6. Let $(K_t)_{t \in \mathbb{R}_{>0}}$ be a family of L^1 convolution kernels, and let $\xi \to m(\xi, t)$ be the Fourier transform of K_t. Suppose that the functions $u \to \frac{m(\xi,u)}{u}$ satisfy the regularity of (5.6). If there exists an $\alpha \in (\frac{1}{2}, 1)$ such that

$$\sup_{\xi \in \mathbb{R}^d \setminus 0} \|t \to m(\xi, t)\|_{L^2_\alpha} = \left(\int_0^\infty \left| t^{\alpha+1} D_t^\alpha \left(\frac{m(\xi, t)}{t} \right) \right|^2 \frac{dt}{t} \right)^{1/2} < C_\alpha < \infty.$$

Then we have that

$$\|\sup_{t>0} K_t * f\|_{L^2(\mathbb{R}^d)} \leq C \|f\|_{L^2}.$$

Proof. Since g_ξ satisfies the regularity conditions of (5.6), we can apply the integration-differentiation identity of Claim 2 to get that

$$\frac{m(\xi, t)}{t} = \frac{1}{\Gamma(\alpha)} \int_t^\infty (u - t)^{\alpha - 1} D_u^\alpha \left(\frac{m(u, \xi)}{u} \right) du.$$

Now for $f \in \mathcal{S}(\mathbb{R}^d)$, we have that

$$\begin{aligned}
(K_t * f)(x) &= \int_{\mathbb{R}^d} m(\xi, t) \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi \\
&= \frac{1}{\Gamma(\alpha)} \int_t^\infty t(u - t)^{\alpha-1} \int_{\mathbb{R}^d} D_u^\alpha \left(\frac{m(\xi, u)}{u} \right) \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi du \\
&= \frac{1}{\Gamma(\alpha)} \int_t^\infty \frac{t}{u} \left(1 - \frac{t}{u} \right)^{\alpha - 1} (P_u^\alpha f)(x) \frac{du}{u},
\end{aligned} \tag{5.10}$$

where we define for $u > 0$,

$$(P_u^\alpha f)(x) = \int_{\mathbb{R}^d} u^{\alpha+1} D_u^\alpha \left(\frac{m(\xi, u)}{u} \right) \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

It is clear from the definition that $P_u^\alpha f$ is a fourier multiplier with symbol

$$p_u^\alpha(\xi) = u^{\alpha+1} D_u^\alpha \left(\frac{m(\xi,u)}{u}.\right)$$

We use Cauchy-Schwartz to get that

$$|K_t * f(x)| \leq \left(\int_t^\infty \frac{t^2}{u^2}\left(1-\frac{t}{u}\right)^{2\alpha-2}\frac{du}{u}\right)^{\frac{1}{2}}\left(\int_0^\infty |P_u^\alpha f(x)|\frac{du}{u}\right)^{\frac{1}{2}}.$$

Using the fact that $2\alpha - 2 > -1$ (since $\alpha > \frac{1}{2}$), a simple change of variables $y = \frac{t}{u}$ yields that the first integral on the right is bounded by a constant only dependent on α. Since the integral $\int_0^\infty |P_u^\alpha f(x)|dx$ is independent of t, we get that

$$\sup_{t>0} |(K_t * f)(x)|^2 \lesssim_\alpha \int_0^\infty |P_u^\alpha f(x)|^2 \frac{du}{u}.$$

Taking L^2 norms finally gives us that

$$\begin{aligned}\left\|\sup_{t>0}|(K_t * f)(x)|\right\|_{L^2}^2 \leq \int_{\mathbb{R}^d}\int_0^\infty |P_u^\alpha f(x)|^2\frac{du}{u}dx = \int_0^\infty \|P_u^\alpha f(x)\|_{L^2}^2\frac{du}{u} = \int_0^\infty \|\widehat{P}_u^\alpha f(x)\|_{L^2}^2\frac{du}{u} \\ = \int_{\mathbb{R}^d}\int_0^\infty \left|u^{\alpha+1}D_u^\alpha\left(\frac{m(\xi,u)}{u}\right)\widehat{f}(\xi)\right|^2\frac{du}{u}d\xi.\end{aligned}$$

But the above term in the absolute value was assumed to have a finite supremum C_α over $|\xi| > 0$. Throwing away the origin of $\mathbb{R}^d$ in the integral (as it's Lebesgue measure zero) and applying Parseval once more, we obtain that $\sup_{t>0} |(K_t * f)(x)|$ is L^2 bounded by C_α. □

We now state an L^p analogue of the preceding lemma in a slightly more restricted setting.

Lemma 7. Let K be an L^1 convolution kernel with Fourier transform $m(\xi)$, and consider the family of operators $(K_{(t)})_{t>0}$. Suppose the function $u \to \frac{m(u\xi)}{u}$ satisfies the regularity of (5.6). For a fixed $p \in (1,\infty)$ if both $m(\xi)$ and $(\xi\cdot\nabla)^\alpha m(\xi)$ have bounded $L^p(\mathbb{R}^d)$ multiplier norm for some $\alpha \in (\frac{1}{p}, 1)$, then we have that

$$\|\sup_{1\leq t\leq 2} K_{(t)} * f\|_{L^p} \leq C\|f\|_{L^p},$$

where C is a function of the multiplier norms of $m(\xi)$ and $(\xi\cdot\nabla)^\alpha m(\xi)$.

Proof. As the family $(K_{(t)})_{t>0}$ satisfies the critera of the previous Lemma, (5.10) still holds. Rather than apply Cauchy-Schwartz, we apply apply Holder's with p and it's conjugate p' and apply the substitution $v = \frac{u}{t}$:

$$\begin{aligned}|K_{(t)} * f| &\lesssim_\alpha \left(\int_t^\infty \frac{t^{p'}}{u^{p'}}\left(1-\frac{t}{u}\right)^{p'(\alpha-1)}du\right)^{1/p'}\left(\int_t^\infty |P_u^\alpha f(x)|^p\frac{du}{u^p}\right)^{1/p} \\ &\leq t^{1/p'}\left(\int_1^\infty v^{-p'\alpha}(v-1)^{p'(\alpha-1)}dv\right)^{1/p'}\left(\int_1^\infty |P_u^\alpha f(x)|^p\frac{du}{u^p}\right)^{1/p}.\end{aligned}$$

Since $\alpha \in (\frac{1}{p}, 1)$, we have that $(\alpha-1)p' > -1$, and since $p < \infty$, we have that $q > 1$. Therefore, the integral on the left converges. The integral on the right is independent of t, and $t^{1/p'}$ is bounded on $t \in [1,2]$, so we get that

$$\begin{aligned}\|\sup_{1\leq t\leq 2}|K_{(t)} * f|\|_{L^p}^p &\lesssim_{\alpha,p} \left\|\left(\int_1^\infty |P_u^\alpha f(x)|^p\frac{du}{u^p}\right)^{\frac{1}{p}}\right\|_p^p \\ &= \int_1^\infty \|P_u^\alpha f\|_p^p\frac{du}{u^p} \lesssim_p \sup_{u\geq 1}\|P_u^\alpha f\|_p^p.\end{aligned}$$

We now bound $P_u^\alpha f$ in terms of the L^p multipliers norms of m and $(\xi \cdot \nabla)^\alpha m$. It follows immediately from the computation in (5.9) that $p_u^\alpha(\lambda\xi) = p_{\lambda u}^\alpha(\xi)$. Since the L^p norms of Fourier multipliers stay the same when the symbol is dilated, to bound the L^p multiplier norm of p_u^α, it suffices to bound the L^p multiplier norm of just p_1^α. Using an integration by parts, it follows that

$$p_1^\alpha(\xi) = D_u^\alpha(m(u\xi)/u)|_{u=1} = \alpha D_u^{\alpha-1}(m(u\xi)/u)|_{u=1} + D_u^\alpha\left(m(u\xi)\right)|_{u=1}.$$

To understand $\alpha D_u^{\alpha-1}(m(u\xi)/u)|_{u=1}$, we apply definition (5.8) and integrate by parts in reverse to get that

$$\alpha D_u^{\alpha-1}(m(u\xi)/u)|_{u=1} = \frac{\alpha}{\Gamma(1-\alpha)}\int_1^\infty (t-1)^{-\alpha}\left(m\frac{(t\xi)}{t}\right)dt = \frac{\alpha}{\Gamma(1-\alpha)}\int_1^\infty (t-1)^{\alpha-2} m(t\xi)dt.$$

Since $\int_1^\infty (t-1)^{\alpha-2} m(t\xi)dt$ converges, we get by Lemma 1 that the L^p multiplier norm of $\alpha D_u^{\alpha-1}(m(u\xi)/u)|_{u=1}$ is bounded by the L^p multiplier norm of $m(\xi)$. Furthermore, $D_u^\alpha\left(m(u\xi)\right)|_{u=1}$ is exactly $(\xi\cdot\nabla)^\alpha m(\xi)$. Therefore, if both $m(\xi)$ and $(\xi\cdot\nabla)^\alpha m(\xi)$ have bounded L^p multiplier norm, by the triangle inequality we conclude that $\sup_{1\le t\le 2}|K_{(t)} * f|$ is L^p bounded as a function of these two multiplier norms. □

5.3 Condition 3 - Strong L^2 boundedness

We now turn to the strong L^2 boundedness of the family of operators T_{jv} as in (5.1) with respect to the Littlewood-Paley decomposition R_j defined in (5.2). We would like to apply Lemma 6 to the family of Fourier multipliers $T_{jv}R_{j+k}$ for a fixed k, and show that the constants returned by the Lemma decay sufficiently as $|k| \to \infty$. In order to apply this Lemma, we need to parametrize the multipliers of $T_{jv}R_{j+k}$ in the variable $t = 2^j v$. Therefore, for a fixed k, we consider the family of Fourier multipliers $K_{k,t}$, and define $\xi \to m_k(\xi, t)$, the Fourier transform of $K_{k,t}$, as

$$m_k(\xi,t) = m(t\xi)(\widehat{P}_{L_{[2^{j(t)+k}]}} - \widehat{P}_{L_{[2^{j(t)+k+1}]}})(\xi),$$

where $j(t) = \lfloor \log_2(t)\rfloor$ and m is the Fourier transform of χ_B. Unfortunately, these symbols are not continuous in t, which is needed to satisfy the regularity conditions (5.6) required for Lemma 6 and conclude strong L^2 boundedness. We instead work with the following family $N_{k,t}$ of Fourier multipliers with symbols

$$n_k(\xi,t) = m(X(t)\xi)(\widehat{P}_{L_{[2^k Y(t)]}} - \widehat{P}_{L_{[2^{k+1} Y(t)]}})(\xi),$$

where we define the functions

$$X(t) = \begin{cases} 2^j + 2(t-2^j) & 2^j \le t \le \frac{2^j+2^{j+1}}{2} \\ 2^{j+1} & \frac{2^j+2^{j+1}}{2} \le t \le 2^{j+1}\end{cases} \qquad Y(t) = \begin{cases} 2^j & 2^j \le t \le \frac{2^j+2^{j+1}}{2} \\ 2^j + 2(t - \frac{2^j+2^{j+1}}{2}) & \frac{2^j+2^{j+1}}{2} \le t \le 2^{j+1}\end{cases}$$

It is clear that the family $K_{k,t}$ contains $N_{k,t}$, as illustrated the figure below. Therefore, if we can show that for any $\alpha \in (\frac{1}{2}, 1)$, that

$$\sup_{\xi\in\mathbb{R}^d\setminus 0} \|t \to n_k(\xi,u)\|_{L_\alpha^2}^2 = \sup_{\xi\in\mathbb{R}^d\setminus 0}\int_0^\infty \left|u^{\alpha+1} D_u^\alpha\left(\frac{n_k(\xi,u)}{u}\right)\right|^2 \frac{du}{u} < \infty,$$

then by Lemma 6 we can conclude that $T_{jv}R_{j+k}f$ is L^2 bounded by some constant depending in k.

In fact, since $n_k(t, 2\xi) = n_k(2\xi, t)$ (because X satisfies $X(2t) = 2X(t)$ and so does Y) and the L_α^2 norm is invariant under scaling, it is enough to show that

$$\sup_{|\xi|\in[1,2]} \|u \to n_k(\xi,u)\|_{L_\alpha^2}^2 = \sup_{|\xi|\in[1,2]}\int_0^\infty \left|u^{\alpha+1} D_u^\alpha\left(\frac{n_k(\xi,u)}{u}\right)\right|^2 \frac{du}{u} < \infty. \tag{5.11}$$

Our strategy to obtain (5.11) is to apply the bound on D^α given by Claim 3, for which we first need a good understanding of $\frac{n_k(\xi,u)}{u}$. We make some preliminary estimates first. For notational convenience, we define $p(\xi) = \widehat{P}_L(\xi) - \widehat{P}_L(2\xi) = e^{-2\pi L|\xi|} - e^{-2\pi L|\xi|}$. It is easy to explicitly check that for θ any unit vector in $\mathbb{R}^d$ that

$$p(u\theta) \lesssim \min(u, u^{-1}) \qquad \frac{d}{du}p(u\theta) \lesssim \min(1, u^{-1}). \tag{5.12}$$

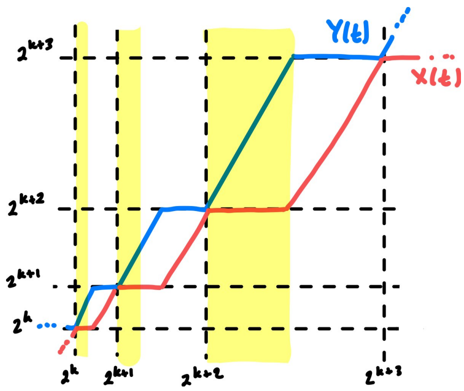

Figure 5: A graph of $X(t)$ and $Y(t)$. The highlighted region are the areas where $N_{k,t}$ are contained in the family $K_{k,t}$, and the remaining regions provide the continuity needed to get between each highlighted region.

By the estimates (4.9) and (4.10) from the previous section, we get that $m(u\theta)$ decays like u^{-1} and can grow at most at a linear rate. Therefore, we have that

$$m(u\theta) \lesssim \min(u, u^{-1}) \qquad \frac{d}{du} m(u\theta) \lesssim \min(1, u^{-1}). \tag{5.13}$$

For all $|\xi| \in [1, 2]$, we note that $u \le |X(u)\xi| \le 4u$ and $\frac{u}{2} \le |Y(u)\xi| \le 2u$. Then by (5.13), we have that uniformly in ξ such that $|\xi| \in [1, 2]$,

$$m(X(u)\xi) \lesssim \min(u, u^{-1}) \qquad \frac{d}{du} m(X(u)\xi) \lesssim \min(1, u^{-1}). \tag{5.14}$$

By the same reasoning, (5.12) tells us that uniformly in ξ such that $|\xi| \in [1, 2]$,

$$p(Y(u)\xi) \lesssim \min(u, u^{-1}) \qquad \frac{d}{du} p(Y(u)\xi) \lesssim \min(1, u^{-1}). \tag{5.15}$$

We introduce the notation $\phi(u) = m(X(u)\xi)$ and $\psi(u) = p(Y(u)\xi)$, where we omit any independence in ξ in ϕ and ψ since all bounds we use are uniform in ξ such that $|\xi| \in [1, 2]$. As the L^2_α norm is invariant under dilation, we note that

$$\phi(u)\psi(2^k u) = \|n_k(\xi, u)\|_{L^2_\alpha} \qquad \|n_{-k}(\xi, u)\| = \|n_{-k}(\xi, 2^k u)\|_{L^2_\alpha} = \phi(2^k u)\psi(u). \tag{5.16}$$

Since our bounds (5.14) and (5.15) for ϕ and ψ are identical, the observation above allows us to restrict to the case where $k \ge 0$. Using (5.14) and (5.15), we obtain the following control over $\frac{n_k(\xi,u)}{u}$ and $\frac{d}{du}\frac{n_k(\xi,u)}{u}$ in the given regions:

	$u \in (0, 2^k]$	$u \in [2^{-k}, 1]$	$u \in [1, \infty)$
$\frac{n_k(\xi,u)}{u}$	$\lesssim 2^k u \lesssim 2^{-k} u^{-1}$	$\lesssim 2^{-k} u^{-1}$	$\lesssim 2^{-k} u^{-3}$
$\frac{d}{du}(\frac{n_k(\xi,u)}{u})$	$\lesssim 2^{k+1} \lesssim u^{-1}$	$\lesssim u^{-1} + 2^k u^{-2} \lesssim u^{-1}$	$\lesssim 2^{-k} u^{-3} + u^{-3} \lesssim u^{-3}$

In this way, what we see is that $\frac{n_k(\xi,u)}{u} \lesssim 2^{-k} \min(u^{-1}, u^{-3})$, and $\frac{d}{du}(\frac{n_k(\xi,u)}{u}) \lesssim \min(u^{-1}, u^{-3})$. Therefore, $\frac{n_k(\xi,u)}{u}$ is both decreasing and concave in u, and we can apply the fractional derivative bound of Lemma 3 to get that

$$D^\alpha_u \frac{n_k(\xi, u)}{u} \lesssim 2^{-(1-\alpha)k} \min(u^{-1}, u^{-3}).$$

Putting everything together, for all $k \in \mathbb{Z}$ we have that

$$\|n(\xi, u)\|^2_{L^2_\alpha} = \int_0^\infty |u^{\alpha+1} D^\alpha h_k(u)|^2 \frac{du}{u} \lesssim 2^{-2(1-\alpha)|k|} \left(\int_0^1 (u^{\alpha+1} u^{-1})^2 \frac{du}{u} + \int_1^\infty (u^{\alpha+1} u^{-3})^2 \frac{du}{u} \right).$$

For $\alpha \in (\frac{1}{2}, 1)$, both integrals on the previous line converge, and so we get that $\|u \to n_k(\xi, u)\|_{L^2_\alpha} \lesssim_\alpha 2^{-(1-\alpha)|k|}$. From the preceding analysis it is clear that $u \to n_k(\xi, u)$ satisfies the regularity conditions of (5.6), so applying Carbery's Lemma 6 gives us

$$\|\sup_{t>0} N_{k,t}\|_{L^2_\alpha} \lesssim_\alpha 2^{-(1-\alpha)|k|}\|f\|_{l^2}.$$

Therefore, we may finally conclude that independent of dimension,

$$\|\sup_{jv} T_{jv} R_{j+k} f\|_{L^2} \lesssim_\alpha 2^{-(1-\alpha)|k|}\|f\|_{l^2}.$$

As $\sum_{k=-\infty}^{\infty}(2^{-(1-\alpha)|k|})^t < \infty$ for all $t \in (0,1]$, we conclude that the family of operators T_{jv} is strong L^2 bounded independent of dimension.

5.4 Condition 4 - Interpolation of Analytic Families of Operators

In this section, we work to show that our family of operators T_{jv} from (5.1) satisfies Condition 4 of Carbery's interpolation theorem independent of dimension. To show this, we look to use Lemma 7. For a $q \in (1,2)$, suppose we can find an $\alpha \in (\frac{1}{q}, 1)$ such that for the multiplier $m_1(\xi)$ corresponding to $(\chi_B - P_L) * f$, the Fourier multipliers with symbols $m_1(\xi)$ and $(\xi \cdot \nabla)^\alpha m_1(\xi)$ are both L^q bounded. We note that $u \to \frac{m_1(\xi u)}{u}$ satisfies the necessary regularity conditions of (5.6) by (5.13) and (5.12), so we can conclude by the Lemma that

$$\|\sup_{t\in[1,2]} (K - P_L)_{(t)} * f\|_{L^q} \leq C_q \|f\|_{L^q}.$$

This C_q is also independent of the dimension $\mathbb{R}^d$. Since the L^p norm of a Fourier multiplier is invariant under dilation of the symbol, we conclude that for the same C_q,

$$\sup_j \|\sup_v (K - P_L)_{(2^j v)} * f\|_{L^p} \leq C_q \|f\|_{L^p}.$$

This is exactly Condition 4, as we hoped for. It is clear that $m_1(\xi)$ is L^p multiplier bounded independent of dimension for all $p \in (1, \infty)$ by Young's inequality and the fact both χ_B and P_L have L^1 norm 1 when acting on $\mathbb{R}^d$ for all n. What is left to show is that $(\xi \cdot \nabla)^\alpha m(\xi)$ is L^p multiplier bounded as well with the precise requirements of p, α as stated in Lemma 7. To do this, we will employ analytic interpolation of operators, stated precisely in Theorem 3. This approach is performed by Carbery, but details are omitted in his paper, so we mainly follow Section 7 of [LM18].

We first try to interpolate between $2 \to 2$ and $r \to r$ estimates for $r < p < 2$ of the family of multiplier operators T_z for $z \in \mathbb{C}$ with symbols $m_z(\xi) = (\xi \cdot \nabla)^z m_1(\xi)$. This approach happens to not work, but it is instructive to see precisely how it fails. We first consider the L^2 multiplier norm of $(\xi \cdot \nabla)^z m_1(\xi)$, which we know is bounded by (in fact, equal to) the supremum over $\xi \in \mathbb{R}^d$ of $(\xi \cdot \nabla)^z m_1(\xi)$ by Parseval. On the line $z = v_1 + iy$, fixing $v_1 \in (0,1)$ and varying y, we have that

$$\begin{aligned} |(\xi \cdot \nabla)^{v_1+iy} m_1(\xi)| &= \left| -\frac{1}{\Gamma(1 - v_1 - iy)} \int_1^\infty (u-1)^{-v_1-iy} \frac{d}{du}(m_1(u\xi))du \right| \\ &= \left| \frac{1}{\Gamma(1 - v_1 - iy)} \right| \int_1^\infty |(u-1)^{-v_1-iy}||u\xi \cdot \nabla m_1(u\xi)| \frac{du}{u}. \end{aligned}$$

Now we recall that $|u\xi \cdot \nabla m_1(u\xi)|$ is uniformly bounded in u and ξ by a constant C, from the estimate (4.11) in Bourgain's L^2 argument and the fact that $|u\xi \cdot \nabla \widehat{P}_L(u\xi)|$ is uniformly bounded. Continuing, we have

$$\begin{aligned} &\left| \frac{1}{\Gamma(1 - v_1 - iy)} \right| \int_1^\infty |(u-1)^{-v_1-iy}||u\xi \cdot \nabla m_1(u\xi)| \frac{du}{u} \\ \leq C &\left| \frac{1}{\Gamma(1 - v_1 - iy)} \right| \int_1^\infty (u-1)^{-v_1-1} du \lesssim_{v_1} 2(\sqrt{1+y^2})^{\frac{1}{2}-v_1} e^{\pi|y|/2}. \end{aligned} \tag{5.17}$$

Where the last step uses the well-known bound that for all half-planes of the form $\mathrm{Re}(z) \geq a$, $a > -1$, one has that

$$\left|\frac{1}{\Gamma(z)}\right| \leq 2(\sqrt{1+\mathrm{Im}(z)^2})^{\frac{1}{2}-a} e^{\pi|\mathrm{Im}(z)|/2}. \tag{5.18}$$

A large concern is at hand: suppose that we are able to sufficiently control the L^r operator norm of T_α on some line $v_0 + iy$ in the complex plane. To apply analytic interpolation, we must be able to control $\log\left|\int_{\mathbb{R}^d} T_z(f)g\right|$ for $f, g \in \mathcal{S}(\mathbb{R}^d)$ on all vertical lines $x + iy$ for $x \in (v_0, v_1)$. We would like to like to be able to naively control this quantity by Cauchy-Schwartz:

$$\left|\int_{\mathbb{R}^d} T_{x+iy}(f)g\right| \leq \|T_{x+iy}\|_{2\to 2}\|f\|_{L^2}\|g\|_{L^2}. \tag{5.19}$$

However, $\|T_{x+iy}\|_{2\to 2} = \sup_\xi (\xi\cdot\nabla)^{x+iy} m_1(\xi)$ is unbounded on the imaginary axis, since the integral of u in (5.17) diverges when we replace v_1 with x. What we will see is that to get good L^r estimates on T_z, we actually need to look on a line *to the left of the imaginary axis*, which we can't do at the moment. To get around this issue, we need a way to "truncate" the integral in u in (5.17).

We consider a new fractional integral, the *fractional Riesz integral with basepoint 2*, for $t \leq 2$ and $\omega \in \mathbb{C}$, $\mathrm{Re}(\omega) > 0$ as

$$i^w f(t) = \frac{1}{\Gamma(\omega)} \int_t^2 (u-t)^\omega f(u)du. \tag{5.20}$$

Just as we defined the fractional derivative $D^z f$ from $I^\omega(f)$, we define a new fractional derivative $d^z(f)(t)$ for $t \leq 2$ and $\mathrm{Re}(z) < 1$ by integrating (5.20) by parts and defining $d^z(f)(t) = i^{-z} f(t)$. Precisely, we have that

$$d^z f(t) = \frac{(2-t)^{-z} f(2)}{\Gamma(1-z)} - \frac{1}{\Gamma(1-z)} \int_t^2 (u-t)^{-z} f'(u)du. \tag{5.21}$$

These new definitions are particularly useful because of the following observation:

Lemma 8. Let $m_1(\xi)$ be the multiplier of $(K - P_L) * f$. Then for $\alpha \in (0,1)$, $(\xi\cdot\nabla)^\alpha m(\xi) - d_t^\alpha m_1(t\xi)|_{t=1}$ is a multiplier on $L^p(\mathbb{R}^d)$ with norm bounded independent of dimension.

Proof. When $-1 < \mathrm{Re}(z) < 0$, we have that $D^z(t) = I^{-z}(t)$ and $d^z(t) = i^{-z}(t)$ by analytic continuation. In this range we see that for $f \in \mathcal{S}(\mathbb{R}^d)$,

$$D^z(f)(t) - d^z(f)(t) = I^{-z}(f)(t) - i^{-z}(f)(t) = \frac{1}{\Gamma(-z)} \int_2^\infty (u-t)^{-z-1} f(u)du. \tag{5.22}$$

Integrating by parts allows us to analytically continue this identity. For $0 < \mathrm{Re}(z) < 1$, we have that

$$D^z(f)(t) - d^z(f)(t) = -\frac{(2-t)^{-z} f(2)}{\Gamma(1-z)} - \frac{1}{\Gamma(1-z)} \int_2^\infty (u-t)^{-z} f'(u)du.$$

When $0 < \mathrm{Re}(z) < 1$, we have that $(\xi\cdot\nabla)^\alpha m_1(\xi) = D_t^\alpha m(t\xi)|_{t=1}$ by Claim 1. Plugging in $\alpha \in (0,1)$ into the expression $(\xi\cdot\nabla)^\alpha m_1(\xi) - d_t^\alpha m_1(t\xi)|_{t=1}$ and integrating by parts in reverse, we get that

$$(\xi\cdot\nabla)^\alpha m_1(\xi) - d_t^\alpha m_1(t\xi)|_{t=1} = \frac{1}{\Gamma(-\alpha)} \int_2^\infty (u-1)^{-\alpha-1} m_1(u\xi)du.$$

Since $\left|\frac{1}{\Gamma(-\alpha)}\int_2^\infty (u-1)^{-\alpha}\right| < 1$ for $\alpha \in (0,1)$, we see by Lemma 1 that the L^p multiplier norm of $(\xi\cdot\nabla)^\alpha m_1(\xi) - d_t^\alpha m_1(t\xi)|_{t=1}$ is bounded by the L^p multiplier norm of $m_1(\xi)$. But this is bounded by 2 independent of dimension by Young's inequality. ☐

In particular, what this allows us to do is apply Carbery's Lemma 7 when for some $p < \infty$ and $\frac{1}{p} < \alpha < 1$, both $m_1(\xi)$ and $d_t^\alpha m_1(t\xi)|_{t=1}$ have bounded L^p multiplier norm, rather than having to work with $(\xi \cdot \nabla)^\alpha m_1(\xi)$. The truncated integral in the definition in $d_t^\alpha m_1(t\xi)|_{t=1}$ from 1 to 2 rather than from 1 to ∞ as in $(\xi \cdot \nabla)^\alpha m_1(\xi)$, will allow us to interpolate past the imaginary axis as we would have liked to earlier.

Now we set up the correct interpolation argument. We choose some $\varepsilon > 0$, and define the family of Fourier multipliers N_z for $z \in \mathbb{C}$ with symbols $n_z(\xi) = d_t^z m(t\xi)|_{t=1}$. On the vertical line $-\varepsilon + iy$, since $\mathrm{Re}(-\varepsilon + iy) < 0$, we may write

$$N_{-\varepsilon+iy}(\xi) = \frac{1}{\Gamma(\varepsilon - iy)} \int_1^2 (u-1)^{\varepsilon - iy - 1} m_1(u\xi) du. \tag{5.23}$$

The L^1 multiplier norm of $m_1(u\xi)$ is bounded by 2 (again by Young's). By the Gamma function bound (5.18), we see that $\left| \frac{1}{\Gamma(\varepsilon - iy)} \int_1^2 (u-1)^{\varepsilon - iy - 1} \right| \le \frac{2}{\varepsilon} (\sqrt{1+y^2})^{\frac{1}{2} - \varepsilon} e^{\pi |y|/2}$, so by an application of Lemma 1 we get that

$$\|N_{-\varepsilon+iy}(\xi)\|_{1\to 1} \le \frac{4}{\varepsilon} (\sqrt{1+y^2})^{\frac{1}{2} - v} e^{\pi|y|/2}.$$

We now analyze the L^2 norm of N_z, bounded by the supremum in ξ of $n_z(\xi)$, on the line $v + iy$ for $v < 1$. We see that

$$|n_{v+iy}(\xi)| = \left| \frac{m_1(2\xi)}{\Gamma(1-v-iy)} - \frac{1}{\Gamma(1-v-iy)} \int_1^\infty (u-1)^{-v-iy} \frac{d}{du}(m_1(u\xi)) du \right|.$$

As discussed before, $|u\xi \cdot \nabla m_1(u\xi)|$ is uniformly bound by some constant, say C_1, in u and ξ. In Bourgain's L^2 paper discussed previously, the quantity α_j corresponding to $\chi_B - P_L$ was shown to be uniformly bounded for $j \in \mathbb{Z}$, so $|m_1(2\xi)|$ is uniformly bounded in ξ by some constant independent of L (and thus B), say C_2. We then get that for v in $[-\varepsilon, 1-\varepsilon]$, using the Gamma bound 5.18,

$$|n_{v+iy}(\xi)| \le C_1 C_2 \left| \frac{1}{\Gamma(1-v-iy)} \right| \int_1^2 (u-1)^{-v-1} du \lesssim 2(\sqrt{1+y^2})^{\frac{1}{2} - v} e^{\pi|y|/2}.$$

Therefore, for $v \in [-\varepsilon, 1-\varepsilon]$ (not just to the left of the imaginary axis!), we see that

$$\|N_{v+iy}\|_{2\to 2} \lesssim 2(\sqrt{1+y^2})^{\frac{1}{2} - v} e^{\pi|y|/2}. \tag{5.24}$$

Setting $v = 1 - \varepsilon$ in the above bound, we see that the family N_z satisfies the necessary growth conditions for analytic interpolation of operators. It is clear that $\int_{\mathbb{R}^d} T_z(f) g$ is analytic in z, so the only thing that remains to be shown before we can apply 3 is the control of $\log \left| \int_{\mathbb{R}^d} T_z(f) g \right|$ for $f, g \in \mathcal{S}(\mathbb{R}^d)$ on all vertical lines $x + iy$ for $x \in (-\varepsilon, 1-\varepsilon)$. Thanks to our modified multipliers, this is immediate by applying the naive Cauchy Schwartz estimate as in (5.19) and plugging in (5.24). The figure below summarizes our estimates, sufficient to apply analytic interpolation of operators.

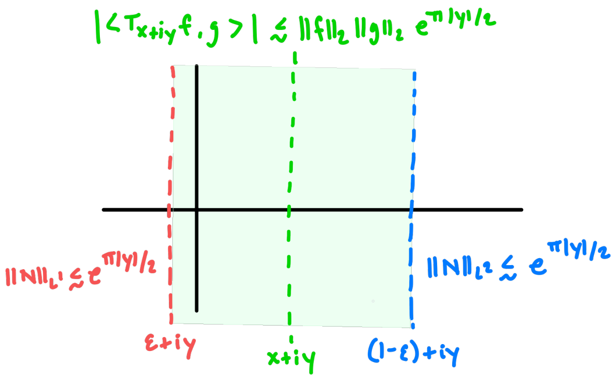

Figure 6: A summary of the bounds needed to apply analytic interpolation

The result of analytic interpolation of operators between L^1 and L^2 bounds tells us that for any $\theta \in [0,1]$ and $\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2} = 1 - \frac{\theta}{2}$, we have that for $\alpha = (1-\theta)(-\varepsilon) + \theta(1-\varepsilon) = \theta - \varepsilon$,

$$\|N_\alpha f\|_{L^p} \lesssim_\alpha \|f\|_{L^p}.$$

Since we want $\alpha > \frac{1}{p}$ to be able to apply Carbery's Lemma 7, we need that $\theta - \varepsilon > 1 - \frac{\theta}{2}$, i.e. $\frac{3}{2}\theta > 1 + \varepsilon$. This in turn forces $p > \frac{3}{2+\varepsilon}$. This, in turn, forces $p > \frac{3}{2}$. This is precisely where the obstruction to $p \in (1, \frac{3}{2}]$ comes in – we need this condition in order for interpolation to work well with Lemma 7.

In summary we get that for any $p > \frac{3}{2}$, we can choose ε small enough so that analytic interpolation of operators gives for some $\alpha > \frac{1}{p}$, that $d_t^z(m_1(t\xi))|_{t=1}$ has bounded L^p multiplier norm independent of dimension. Therefore, the same can be done with $(\xi \cdot \nabla)^\alpha m_1(\xi)$, and we conclude with Lemma 7 that Condition 4 of Carbery's interpolation theorem holds for $p > \frac{3}{2}$.

6 The ℓ^q Balls and Open Questions

As we just saw, Carbery's proof of $L^p(\mathbb{R}^d)$ bounds on M_B independent of d and B hit an obstacle at $p = \frac{3}{2}$ during the interpolation process needed to satisfy Condition 4 of the key interpolation theorem 8. As the rest of Carbery's proof holds for $p > 1$ independent of dimension and B, what can be done to show the family of operators $T_{jv} f = (\chi_B - P_L)_{(2^j v)} * f$ satisfies Condition 4 of Carbery's interpolation theorem for $p \in (1, \frac{3}{2}]$?

One idea is as follows. Continuing with the family of operators N_z and the multiplier m_1 as defined in the previous subsection, suppose that the family N_z satisfies the necessary growth conditions for analytic interpolation of families of operators on the strip $S = \{z : Re(z) \in [-\varepsilon, A]\}$ for arbitrarily large $A \in \mathbb{R}$, rather than just on the strip $\{z : Re(z) \in [-\varepsilon, 1-\varepsilon]\}$. Then it's easy to see that for any fixed $p \in (1, \infty)$, we can choose A large enough such that N_α is L^p bounded for any $\alpha \in (\frac{1}{p}, 1)$, and thus Condition 4 holds by the previous subsection. This idea is promising, since our definition of $d_u^z m(u\xi)|_{u=1}$ can be analytically continued to the entire complex plane. By repeatedly integrating i^{-w} by parts for $\mathrm{Re}(w) > 0$, where i^w is the Riesz integral as in (5.20), for all $k \in \mathbb{Z}$ we get expressions for d^z valid for $\mathrm{Re}(z) < k$ as

$$d^z m_1(\xi) = \sum_{j=0}^{k-1} (-1)^j \frac{(2-t)^{-z+j}}{\Gamma(j+1-z)} \frac{d^{(j)}}{d^{(j)} t} m_1(t\xi)\bigg|_{t=2} + \frac{(-1)^k}{\Gamma(k-z)} \int_t^2 (u-t)^{-z+k-1} \frac{d^{(j)}}{d^{(j)} u} m_1(u\xi) du.$$

In order to control the family of symbols $n_z = d_u^z m_1(u\xi)|_{u=1}$ considered in the proof of Carbery's Condition 4 past $\mathrm{Re}(z) = 1 - \varepsilon$, we need to be able to control $\frac{d^{(j)}}{d^{(j)} u} m_1(u\xi) du$ for $j \leq \lceil A \rceil$, analogous to how we needed an understanding of $m_1(u\xi)$ and $\frac{d}{du} m_1(u\xi)$ in Carbery's case. It isn't hard to show that

$$\frac{d^{(j)}}{d^{(j)} u} m_1(u\xi) \lesssim \frac{|\xi|^j}{1 + |\xi|}.$$

The extra factors of $|\xi|^j$ that are picked up from $(\xi \cdot \nabla)^z m_1(\xi)$ as $\mathrm{Re}(z)$ increases pose a problem to interpolating the family of multipliers $n_z = d_u^z m_1(u\xi)|_{u=1}$ past $\mathrm{Re}(z) = 1 - \varepsilon$. Müller, in his 1990 paper [Mül90], was able to compensate for this issue and satisfy Carbery's Condition 4 by performing analytic interpolation on the strip $S = \{z : Re(z) \in [-\varepsilon, A]\}$ instead with the family of multipliers defined by the symbols

$$m_{z,\varepsilon}(\xi) = (1 + |\xi|)^{1-z-\varepsilon} d_t^z m_1(t\xi)|_{t=1}. \tag{6.1}$$

The detailed analysis involved in bounding $m_{z,\varepsilon}(\xi)$ in L^p multiplier norm for $p < 2$ on the line $z = -\varepsilon + iy$ and bounding $m_z(\xi)$ in L^2 multiplier norm on the entire strip $S = \{z : Re(z) \in [-\varepsilon, A]\}$ (equivalently, bounding $\sup_\xi m_{z,\varepsilon}(\xi)$ on S) involves dependence on particular geometric properties of our fixed convex body. We associate two geometric quantities σ and Q to a symmetric convex body $B \subset \mathbb{R}^d$ of volume 1. We define $\frac{1}{\sigma(B)}$ to be the infimum of the volume of the $d-1$ dimensional cross sections of our body with all hyperplanes crossing through the origin. We define $Q(B)$ to be the supremum of the volume of the $d-1$ dimensional orthogonal projections on all hyperplanes. As $\sigma(B)$ and $Q(B)$ are invariant under

$SL(\mathbb{R}^d)$, we can assume that B is in isotropic position and thus satisfies the conclusion of Lemma 4 so that all $d-1$ dimensional cross sections are the same volume up to a constant. In this way, we see that $\sigma(B)$ is equal to the isotropy constant L of B up to the universal constant in Lemma 4.

With this setup, Müller proved the following:

Theorem 9. For any $p \in (1, \infty)$, $\varepsilon \in (0, \frac{1}{2})$, $\alpha \in (\frac{1}{2}, 1)$, and B a symmetric convex body of volume 1 in $\mathbb{R}^d$, we have that

$$\|m_{\alpha,\varepsilon}\|_{p\to p} \leq C_\alpha(p, \sigma(B), Q(B)).$$

In particular, this constant is independent of d.

In particular, once we fix a p, choosing ε small enough so that $\alpha := 1-\varepsilon > \frac{1}{p}$ gives us that $m_{\varepsilon,1-\varepsilon} = d_t^{1-\varepsilon} m(t\xi)|_{t=1}$ is L^p multiplier bounded. By Lemma 8, we see that $(\xi \cdot \nabla)^{1-\varepsilon} m(\xi)$ is L^p multiplier bounded, and therefore Carbery's Condition 4 is met. Combining Müller's contributions with the rest of Carbery's proof, we get the following result:

Corollary 2. Consider a sequence of symmetric convex bodies $(B_d)_{d\in\mathbb{N}}$ such that $\mathrm{Vol}_d(B_d) = 1$. If the quantities $\sigma(B_d)$ and $Q(B_d)$ are uniformly bounded in d, then for all $p \in (1, \infty)$ the L^p operator norm of maximal functions M_{B_d} can be bounded with constant independent of d.

While we do not show it here, one can show that for $q \in (1, \infty)$ and $(B_d)_{d\in\mathbb{N}}$ the family of ℓ^q balls in $\mathbb{R}^d$, the geometric quantities $Q(B_d)$ and $\sigma(B_d)$ are bounded independent of d. Thus we may conclude that the maximal function for ℓ^q balls is L^p bounded independent of dimension for all $p > 1$.

In the case of the family of cubes, i.e. the family B_d of ℓ^∞ balls in $\mathbb{R}^d$, we have that $Q(B_d) = \sqrt{d}$, and so Müller's argument doesn't allow us to conclude that the maximal function for cubes has dimension-free L^p bounds in the case of $p \in (1, \frac{3}{2}]$. This case was only resolved in 2014, where Bourgain in [Bou14] was able to explicitly show the necessary decay of $\frac{d^{(j)}}{d^{(j)}u} m(u\xi)$ for m the Fourier transform of the cube. This allowed him to perform an analytic interpolation of families argument to show an analogue of Carbery's Condition 4.

We end this article with a short discussion of open questions. If a sequence of symmetric convex bodies $(B_d)_{d\in\mathbb{N}}$ do not have $Q(B_d)$ and $\sigma(B_d)$ bounded uniformly in d, it is still unknown precisely when the family of maximal operators M_{B_d} enjoy L^p operator norm bounds independent of d for $p \in (1, \frac{3}{2}]$. Analogous to the dimension-free L^p estimates to the spherical maximal function, a similar question can also be asked about the behavior of L^p operator norms of maximal functions corresponding to the boundaries of a sequence of convex bodies $(B_d)_{d\in\mathbb{N}}$ (where the surface measure on the sphere is replaced with a normalized Hausdorff measure). Finally, in the introduction, a result of Tišer was mentioned that passed the dimension free bounds on the Euclidean maximal function to the infinite dimensional case, in order to prove a Lebesgue differentiation theorem for certain Gaussian measures on Hilbert spaces. It is natural to ask if the same can be done with the results of Carbery and Müller as well as the recent results of Bourgain for cubes to prove a Lebesgue differentiation theorem for ℓ^q balls in Hilbert spaces as well.

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References

[Ald11] J. M. Aldaz. "The Weak Type $(1, 1)$ Bounds for the Maximal Function Associated to Cubes Grow to Infinity with the Dimension". In: *Annals of Mathematics* 173.2 (2011), pp. 1013–1023.

[Alm19] AlmostOriginality. *Basic Littlewood-Paley theory III: applications*. 2019. URL: https://almostoriginality.wordpress.com/2019/04/19/basic-littlewood-paley-theory-iii/.

[Bou14] J. Bourgain. "On the Hardy-Littlewood Maximal Function for the Cube". In: *Israel Journal of Mathematics* 203 (2014), pp. 275–293.

[Bou86a] J. Bourgain. "On High Dimensional Maximal Functions Associated to Convex Bodies". In: *American Journal of Mathematics* 108.6 (1986).

[Bou86b] J. Bourgain. "On the L^p-bounds for maximal functions associated to convex bodies in $\mathbb{R}^n$". In: *Israel Journal of Mathematics* 54 (1986), pp. 257–265.

[Car86] Anthony Carbery. "An Almost-Orthogonality Principle with Applications to Maximal Functions associated to Convex Bodies". In: *Bulletin of the American Mathematical Society* 14.2 (1986).

[Gra14] Loukas Grafakos. *Classical Fourier Analysis*. Springer, 2014.

[Gra24] Loukas Grafakos. *Fundamentals of Fourier Analysis*. Springer, 2024.

[KL22] Bo'az Klartag and Joseph Lehec. "Bourgain's slicing problem and KLS isoperimetry up to polylog". In: *Geometric and Functional Analysis* 32 (2022), pp. 1134–1159.

[LM18] O. Guédon L. Deleaval and B. Maurey. "Dimension free bounds for the Hardy-Littlewood maximal operator associated to convex sets". In: *Annales de la Faculté des sciences de Toulouse* 27.1 (2018), pp. 1–198.

[Mel03] Antonios D. Melas. "The best constant for centered Hardy-Littlewood maximal inequality". In: *Annals of Mathematics* 157.2 (2003), pp. 647–688.

[Mül90] Detlef Müller. "A Geometric Bound for Maximal Functions associated to Convex Bodies". In: *Pacific Journal of Mathematics* 142.2 (1990).

[SS83] E. M. Stein and J. O. Strömberg. "Behavior of maximal functions in R^n for large n". In: *Arkiv for Matematik* 54 (1983), pp. 259–269.

[Ste70] Elias Stein. *Topics in Harmonic Analysis Related to the Littlewood-Paley Theory*. Princeton University Press, 1970.

[Ste76] E. M. Stein. "Maximal functions: Spherical means". In: *PNAS* 73.7 (1976).

[Ste82] E. M. Stein. "The development of square functions in the work of A. Zygmund". In: *Bulletin of the American Mathematical Society* 7.2 (1982), pp. 359–376.

[Ste83] E. M. Stein. "Some Results in Harmonic Analysis in $\mathbb{R}^n$ for $n \to \infty$". In: *Bulletin of the American Mathematical Society* 9.1 (1983).

[Tao11] Terrence Tao. *Stein's spherical maximal theorem*. 2011. URL: https://terrytao.wordpress.com/2011/05/21/steins-spherical-maximal-theorem/.

[Tiš88] Jaroslav Tišer. "Differentiation theorem for Gaussian measures on Hilbert space". In: *Trans. Amer. Math. Soc.* 308.2 (1988), pp. 655–666. ISSN: 0002-9947,1088-6850. DOI: 10.2307/2001096. URL: https://doi.org/10.2307/2001096.