# A Proof of the Gauss-Bonnet Theorem

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## 1 Introduction

The Gauss Bonnet theorem gives a connection between the local information about a smooth surface, the curvature at a point, and the global information about the surface, the Euler characteristic. More specifically,

> **Theorem 1.** (Gauss-Bonnet) Given a smooth, compact Riemannian 2-manifold $(M, g)$, the Gaussian curvature $K$, and the Euler characteristic $\chi$, we have that
>
> $$\int_M K \, dA = 2\pi\chi(M)$$

For the simplicity of this exposition and the concreteness of ideas, I will be proving this theorem in a three dimensional setting, i.e our Riemannian 2-manifold $(M, g)$ will be regular surface in $\mathbb{R}^3$ (a surface that's locally homeomorphic to $\mathbb{R}^2$ with injective differential). Furthermore, I will be assuming that the surface we are working with is orientable (since any compact nonorientable surface isn't embeddable into $\mathbb{R}^3$)[1].

The strategy of this proof will be to work with a triangulation of our surface, and to prove a local statement that relates the curvature on the interior of a triangle to the angles of the triangle. More specifically, we will work with *geodesic triangles*, triangles whose sides are geodesics, i.e. curves that take the shortest path from one vertex to another.

[1]However, if Gauss-Bonnet is proved in generality for orientable surfaces, it is very easy to prove it for nonorientable surfaces. We just pass to the orientable double cover and apply Gauss-Bonnet there, and from here the result follows quickly for the base surface.

**Theorem 2.** (Local Gauss-Bonnet) Let $\Omega$ be the interior of a geodesic triangle on $M$, and let $\omega_1, \omega_2, \omega_3$ be the exterior angles of this triangle (both to be defined precisely later). Then

$$\int_\Omega K\,dA + \omega_1 + \omega_2 + \omega_3 = 2\pi$$

Triangulating our smooth surface with geodesic triangles and applying the Local Gauss-Bonnet theorem to each triangle in our triangulation will allow us to conclude the global version[2].

To prove Local Gauss-Bonnet, we need to make a connection between the Gaussian Curvature of points in the interior of our geodesic triangle $\Omega$, and the *geodesic curvature* of points on the boundary $\partial\Omega$. We will do this by showing that both of these quantities are intrinsic to the surface, i.e. they can be expressed in terms of the coefficients of the metric and their derivatives, quantities that can be measured without reference to the ambient space the surface is embedded in. A fish in the sea lives in some aquatic ambient space, so its curvature properties may be defined with respect to this ambient space, but these curvature properties remain true of a fish out of water as well.

Much of this exposition follows the presentation of [Car76], where more details on the fundamental forms, curvature, and Gauss-Bonnet in the three-dimensional setting can be found. For a general treatment of Gauss-Bonnet, see, for instance, Chapter 9 of [Lee18].

# 2 A quick review of metrics

Given a regular surface $S \subset \mathbb{R}^3$, an open set $V \subset S$, and a point $p \in V$, the metric, or first fundamental form, on the vector space $T_p(S)$ is a choice of inner product on this vector space. On a surface embedded in $\mathbb{R}^3$, $T_p(S)$ will inherit the standard inner product structure from the ambient space. Given a local parametrization $\varphi : U \to V$ of $S$ with $U \subset \mathbb{R}^2$, we have a basis of $T_p(S)$ given by $\varphi_u$ and $\varphi_v$, the images of the basis vectors $e_1, e_2$ under the Jacobian at $\varphi^{-1}(p)$. Under this basis, we can write our metric as the bilinear form

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}, \text{ where } \begin{cases} E = \langle \varphi_u, \varphi_u \rangle \\ F = \langle \varphi_u, \varphi_v \rangle = \langle \varphi_v, \varphi_u \rangle \\ G = \langle \varphi_v, \varphi_v \rangle \end{cases}$$

This metric clearly changes smoothly as we move $p$ across $V$. With this notation, we will now work to express the Gaussian and geodesic curvatures *intrinsically.* Intuitively, this means that Gaussian and geodesic curvatures can be calculated without reference to the space it's embedded in. Precisely, this means that the Gaussian and geodesic curvatures can be written in terms of $E, F, G$, and their derivatives.

# 3 Gaussian Curvature

In this section, we will define the Gaussian Curvature $K$, and give a rough proof of Gauss's *Theorema Egregium*, that $K$ is an intrinsic property to a surface. As the curvature of a surface, a property relating to the change in normal vector, fundamentally depends on the choice of embedding into some ambient surface, the fact that $K$ can be described intrinsically is surprising.

As mentioned before, throughout the paper we will assume that our surface $S$ is orientable, i.e. there exists a choice of unit normal vector $N(p)$ to each point $p \in S$ such that the function taking $p$ to it's normal vector is globally defined and smooth. We'll say that a coordinate frame $\varphi_u, \varphi_v$ of a neighborhood of $S$ is positively oriented with respect to a given orientation if $\langle \varphi_u \times \varphi_v, N(p) \rangle > 0$ (using the standard cross product in $\mathbb{R}^3$).

$\boxed{\text{this is a box wee}}$

[2]It will also be the case that smooth surfaces can be triangulated with geodesic triangles, as brought up in Section 7

## 3.1 The Gauss Map and the Second Fundamental Form

Given an orientable surface $S$, we can define the Gauss Map $N : S \to S^2$ sending a point $p \in S$ to it's unit normal vector $N(p)$. This is clearly a smooth map (by the definition of orientability), and so we have the differential $dN_p : T_p(S) \to T_{N(p)}(S^2)$. Since the planes $T_p(S)$ and $T_{N(p)}(S^2)$ are parallel (they have the same normal vector), we can view $dN_p$ as a map from $T_p(S)$ to itself. Since $dN_p$ is a linear map describing how the normal vector changes in a small region of $S$, it makes sense that $dN_p$ should capture how much the surface "curves" in a small region. In this way, we define the Gaussian curvature as follows:

**Definition 1.** The **Gaussian curvature** $K$ of $S$ at $p$ is the determinant of $dN_p$.

We make a quick observation:

**Lemma 1.** $dN_p$ is a self-adjoint map.

*Proof.* Let $\varphi(u, v)$ be a local parametrization of $S$ around $p$, so we need to check that $\langle dN_p(\varphi_u), \varphi_v\rangle = \langle \varphi_u, dN_p(\varphi_v)\rangle$. Given a parametrized curve $\alpha(t) = \varphi(u(t), v(t))$ with $\alpha(0) = p$, then

$$dN_p(\alpha'(0)) = \frac{d}{dt} N(u(t), v(t))\Big|_{t=0} = N_u u'(0) + N_v v'(0)$$

From this, we get that $dN_p(\varphi_u) = N_u$ and $dN_p(\varphi_v)$ is $N_v$, so showing that $\langle dN_p(\varphi_u), \varphi_v\rangle = \langle \varphi_u, dN_p(\varphi_v)\rangle$ is equivalent to showing that

$$\langle N_u, \varphi_v\rangle = \langle \varphi_u, N_v\rangle$$

This follows quickly from the fact that $\langle N, \varphi_u\rangle = \langle \varphi_v, N\rangle = 0$. Taking the derivative of $\langle N, \varphi_u\rangle$ relative to $v$, the derivative of $\langle \varphi_v, N\rangle$ relative to $u$, and noting that $\varphi_{uv} = \varphi_{vu}$ by smoothness, we get that

$$\langle N_u, \varphi_v\rangle = -\langle N, \varphi_{uv}\rangle = -\langle \varphi_{vu}, N\rangle = \langle \varphi_u, N_v\rangle$$

This completes the proof. □

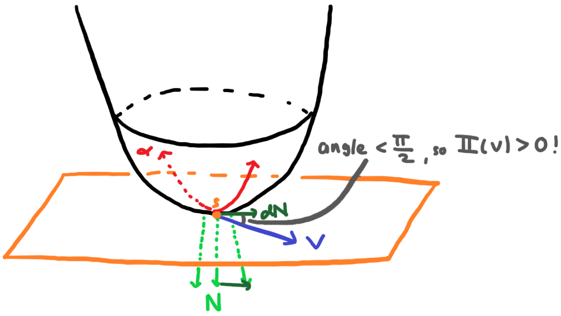
Since $dN_p$ is self-adjoint, we may define the quadratic form $Q(v) = \langle dN_p(v), v\rangle$, leading us to the following definition.

**Definition 2.** The **second fundamental form** $II_p$ of $S$ at $p$ is defined to be the quadratic form $-\langle dN_p(v), v\rangle$, which with respect to local coordinates can be represented with a matrix of the form

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix}$$

(This matrix is symmetric since $dN_p$ is self-adjoint)

Contrary to the first fundamental form, which tells you how distances and angles are computed within the surface (an intrinsic property), the second fundamental form tells us something about how the normal vector changes, an fundamentally extrinsic measurement. In general (not just in $\mathbb{R}^3$), the second fundamental form is measured relative to an embedding into a higher dimensional manifold. When this form is positive definite, i.e. $\langle dN(v), v\rangle \geq 0$ for all $v \in T_p(s)$, this means that our surface is curving "upwards" in all directions from $s$, and that our surface is convex at $s$ as shown in the picture below. This form being negative definite tells us that our surface is concave at $s$, and our form outputting both positive and negative values tells us that our surface looks saddle shaped at $s$.




To work toward giving an intrinsic description of Gaussian curvature, we would like to find the coefficients of the matrix of $dN$ in terms of the first and second fundamental forms. In the process, we will compute explicitly the terms $e, f, g$ of the second fundamental form in terms of local coordinates.

Given a local parameterization $\varphi(u, v)$ of a neighborhood of a point $p \in S$, and a parameterized curve $\alpha(t) = \varphi(u(t), v(t))$ through $p$, we have that $\alpha' = \varphi_u u' + \varphi_v v'$. We also have that

$$dN(\alpha') = N_u u' + N_v v'$$

Then in a basis of our local coordinates $\varphi_u, \varphi_v$, we can write $dN$ in matrix form as

$$dN \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

Where

$$N_u = a_{11}\varphi_u + a_{21}\varphi_v \tag{1}$$
$$N_v = a_{12}\varphi_u + a_{22}\varphi_v \tag{2}$$

From this, we can explicitly compute that the second fundamental form acts on a vector $\alpha' \in T_p(S)$ as

$$II_p(\alpha') = -\langle dN(\alpha'), \alpha' \rangle = -\langle N_u u' + N_v v', \varphi_u u' + \varphi_v v' \rangle = e(u')^2 + 2fu'v' + g(v')^2$$

Where $e, f, g$ are given by (using the fact that $\langle N, \varphi_u \rangle = \langle N, \varphi_v \rangle = 0$)

$$\begin{aligned} e &= -\langle N_u, \varphi_u \rangle = \langle N, \varphi_{uu} \rangle \\ f = -\langle N_v, \varphi_u \rangle = \langle N, \varphi_{uv} \rangle &= \langle N, \varphi_{vu} \rangle = -\langle N_u, \varphi_v \rangle \\ g &= -\langle N_v, \varphi_v \rangle = \langle N, \varphi_{vv} \rangle \end{aligned}$$

Plugging in our expressions for $N_u$ and $N_v$ from (1) and (2) into our expressions of $e, f, g$ above gives us that

$$\begin{aligned} -e &= a_{11}E + a_{21}F \\ -f = a_{11}F + a_{21}G &= a_{12}E + a_{22}F \\ -g &= a_{12}F + a_{22}G \end{aligned}$$

We can encode this information in matrix form as

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

It is clear that by taking inverses (since the first fundamental form is full rank) that the $a_{ij}$ can be explicitly determined. Furthermore, taking determinants on both sides gives us that

$$K = \frac{eg - f^2}{EG - F^2}$$

Which gives us an expression of $K$ in terms of the coefficients of the first and second fundamental forms.

### 3.2 The Christoffel Symbols

At each point in the chart given by our local parametrization $\varphi$, we have an oriented coordinate system of $\mathbb{R}^3$ given by $\{\varphi_u, \varphi_v, N\}$. Therefore, we can express the derivatives of $\varphi_u$ and $\varphi_v$,in terms of $\varphi_u, \varphi_v$ and $N$ as

$$\begin{aligned} x_{uu} &= \Gamma^1_{11}\varphi_u + \Gamma^2_{11}\varphi_v + L_1 N \\ x_{uv} &= \Gamma^1_{12}\varphi_u + \Gamma^2_{12}\varphi_v + L_2 N \\ x_{vu} &= \Gamma^1_{21}\varphi_u + \Gamma^2_{21}\varphi_v + \overline{L_2} N \\ x_{vv} &= \Gamma^1_{22}\varphi_u + \Gamma^2_{22}\varphi_v + L_3 N \end{aligned} \tag{3}$$

These coefficients $\Gamma^i_{jk}$ are called the **Christoffel Symbols** of $S$ in the parameterization $\varphi$. By inner producting each of these equations with $N$, we quickly see that $L_1 = e$, $L_2 = \overline{L_2} = f$, and $L_3 = g$. Furthermore, it is easy to see that the Christoffel symbols are symmetric with respect to the lower indices.

The reason why the Christoffel Symbols are important to us is that once we inner product the equations above with $\varphi_u$ and $\varphi_v$, we get three systems of two equations:

$$\begin{aligned} &\begin{cases} \Gamma^1_{11}E + \Gamma^2_{11}F = \frac{1}{2}E_u, \\ \Gamma^1_{11}F + \Gamma^2_{11}G = F_u - \frac{1}{2}E_v \end{cases} \\ &\begin{cases} \Gamma^1_{12}E + \Gamma^2_{12}F = \frac{1}{2}E_v, \\ \Gamma^1_{12}F + \Gamma^2_{12}G = \frac{1}{2}G_u \end{cases} \\ &\begin{cases} \Gamma^1_{22}E + \Gamma^2_{22}F = F_v - \frac{1}{2}G_v, \\ \Gamma^1_{22}F + \Gamma^2_{22}G = \frac{1}{2}G_u \end{cases} \end{aligned} \tag{4}$$

Each of these systems has determinant $EG - F^2 \neq 0$ since the first fundamental form has full rank, and therefore *the Christoffel Symbols can be expressed completely in terms of the metric*! Anything that can be written in terms of the Christoffel Symbols is therefore intrinsic to the surface.

Given the clear equality $(x_{uu})_v = (x_{uv})_u$ (which holds by smoothness), we use the values of (3) to get that

$$(\Gamma^1_{11}\varphi_u + \Gamma^2_{11}\varphi_v + eN)_u = (\Gamma^1_{12}\varphi_u + \Gamma^2_{12}\varphi_v + fN)_u$$

Using the chain rule and (3) again, and equating the $\varphi_v$ terms (which we can do since $\varphi_u, \varphi_v, N$ is a basis for our space) gives us that

$$\Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} + ea_{22} + (\Gamma^2_{11})_v = \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{12} + fa_{21} + (\Gamma^2_{12})_u$$

Where $a_{11}$ and $a_{12}$ are as defined in (1) and (2). As mentioned before, these coefficients can be explicitly solved, and plugging in these values and rearranging gives us that

$$(\Gamma^2_{12})_u - (\Gamma^2_{11})_v + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{11}\Gamma^2_{12} = -E\frac{eg - f^2}{EG - F^2} = -EK$$

This expression allows us to conclude Gauss's *Theorema Egregium*, that the Gaussian curvature is in fact intrinsic to the surface itself. If we use (4) to solve explicitly for the Christoffel symbols, plug them into the above equation, and pick $\varphi$ to be an orthogonal parametrization of our neighborhood of $s$ (so that $\langle \varphi_u, \varphi_v \rangle = 0$) to simplify our result, we get the celebrated Gauss Formula:

**Theorem 3.** (Gauss Formula) If $\varphi$ is an orthogonal parametrization of a neighborhood of $p$ (i.e. $F = 0$), then the Gaussian curvature $K$ at $p$ can be expressed as

$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right]$$

# 4 Geodesic Curvature

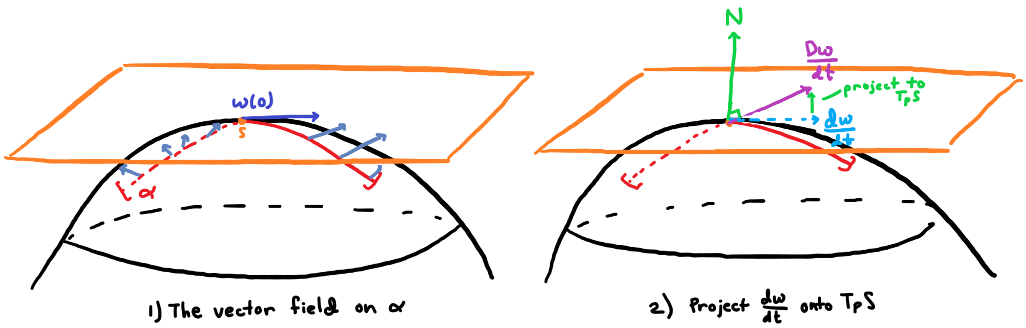
While the Gaussian curvature is the only curvature that shows up in the Gauss-Bonnet formula, the way we will understand the integral of the Gaussian curvature on the interior of a triangle is by relating it to properties of its boundary. One of the properties that will arise in this connection is the *geodesic curvature* of the boundary, a quantity that measures the curvature of a parameterized curve rather than a 2-dimensional area of the surface. However, we will completely understand the geodesic curvature (in fact, it's just zero) if we choose the edges of the triangle to be geodesics on our surface. We will begin by defining geodesic curvature in terms of the ambient space, and then finding an intrinsic way to express this quantity.

## 4.1 The Covariant Derivative and Geodesics

The covariant derivative gives us a generalization of the differentiation of vector fields in Euclidean space.

**Definition 3.** Let $S$ be a regular surface and $U$ an open set of $S$, with $p \in U$. Let $\alpha : (-\varepsilon, \varepsilon) \to U$ be a regular curve (smooth with injective differential) with $\alpha(0) = p$, and consider some $y \in T_p(S)$. Given a smooth vector field $w$ along $\alpha$, we define the **covariant derivative** relative to the vector $y$ as the projection of $\frac{dw}{dt}$ onto $T_p(S)$. We denote this as $(D_y w)(p)$ or $\frac{Dw}{dt}(0)$. Note that $w$ being a vector field along $\alpha$ means $w(t)$ maps to a tangent vector in $T_{\alpha(t)}S$, and smoothness means that for a smooth coordinate frame $\varphi_u, \varphi_v$, $w(t) = a(t)\varphi_u + b(t)\varphi_v$ with $a, b$ smooth.

What the covariant derivative is doing is picking out the part of the change in our vector field along our parameterized curve that lies in the tangent plane of our surface. An attempt at a picture is given below if it helps for visualization. For a potentially better picture, see page 238 of [Car76].




It is not too hard to show that the covariant derivative is independent of the curve $\alpha$ and only dependent on $y$ by showing that it can be expressed in terms of the Christoffel symbols.

If we pick our smooth vector field $w$ to be the tangent vector field $\alpha'(t)$ and interpret $\alpha(t)$ as the path of a particle, then $\frac{D\alpha'(t)}{dt}$ is the tangential component of the acceleration of the particle, i.e how much the particle is accelerating from the point of view on the surface. For an explicit example, consider a cylinder embedded in $\mathbb{R}^3$, and a circular path around it. The centripetal acceleration needed to keep a particle on this circular trajectory points normal to the surface, so in this case, $\frac{D\alpha'}{dt}$ is zero, and the particle doesn't accelerate on the surface.

Our circular path on our cylinder happens to be a particular example of a geodesic.

**Definition 4.** A parameterized curve $\alpha \in S$ is a **geodesic** if $\frac{D\alpha'}{dt} = 0$ for all points on the curve $\alpha$.

With a little bit of work, it can be seen that this definition forces $\alpha$ to be parametrized proportional to arclength. Then a geodesic curve is one where a particle with constant velocity experiences no acceleration on the surface, so a geodesic curve between two points $a$ and $b$ is the path of shortest length between these two points.

Given these definitions, we'll define the geodesic curvature as follows. We consider a regular curve $C$ on an oriented surface $S$, and $\alpha(s)$ an arclength parametrization of $C$ with $\alpha(0) = p$. Therefore, the tangent vector field on $\alpha$ is a unit vector field, and $\frac{d\alpha'}{ds}(s)$ is normal to $\alpha(s)$. Since $\frac{D\alpha'}{ds}(0)$ is also perpendicular to the unit normal vector $N$ at the point $p$, we can express

$$\frac{D\alpha'}{dt} = \lambda(N \times \alpha'(t))$$

For some real coefficient $\lambda$.

**Definition 5.** We denote this $\lambda$ as either $\left[\frac{D\alpha'}{dt}\right]$ or $k_g$, as the **(signed) geodesic curvature** of $C$ at $p$.

From a point of view external to the surface the absolute value of $k_g$ is the norm of $\alpha''(0)$ projected onto $T_p(S)$, i.e. how much a particle on $\alpha$ is accelerating on the surface. From this, we immediately see that the geodesic curvature along a geodesic is zero, and that the geodesic curvature is a measurement of how far a curve is from being a geodesic.

More generally, given $\alpha$ a regular curve, and $w$ any smooth unit vector field on $\alpha$, we have that $\frac{Dw}{dt} = \lambda(N \times w(t))$ for some real coefficient $\lambda$ for the same reasons as before.

**Definition 6.** The **algebraic value of the covariant derivative** $\frac{Dw}{dt}$, denoted by $\left[\frac{Dw}{dt}\right]$, is the real number $\lambda$.

To be able to find an intrinsic description of the geodesic curvature, we will more generally find an intrinsic definition of the algebraic value of the covariant derivative.

## 4.2 An Intrinsic Description

To complete our task, we will need a way to measure the angle between two unit vector fields $v$ and $w$ along a parametrized curve $\alpha : I \to S$. To do this, consider $\overline{v}$ to be the unique unit vector field such that $\{v, \overline{v}\}$ is a positively oriented orthonormal basis along $\alpha$ (given by the Gram-Schmidt process, which is smooth), and therefore, we may write $w(s) = a(s)v(s) + b(s)\overline{v}(s)$ for $a, b$ smooth functions from $I \to \mathbb{R}$ such that $a^2 + b^2 = 1$. We define the angle $\theta(t)$ between $v$ and $w$ to be such that $\cos(\theta) = a(t)$ and $\sin(\theta) = b(t)$, which is clearly smooth as well.

**Lemma 2.** Given two smooth unit vector fields $v$ and $w$ along a curve $\alpha : I \to S$, we have that

$$\left[\frac{Dw}{dt}\right] - \left[\frac{Dv}{dt}\right] = \frac{d\theta}{dt}$$

*Proof.* As before, we may write $w = v\cos\theta + \overline{v}\sin\theta$. Letting $\overline{w} = N \times w$, we see that $\overline{w} = -v\sin\theta + \overline{v}\cos\theta$. Differentiating our expression for $w$ with respect to $t$ gives us that

$$w' = -(\sin\theta)\theta' v + (\cos\theta)v' + (\cos\theta)\theta'\overline{v} + (\sin\theta)\overline{v}'$$

Taking the inner product of $w'$ with $\overline{w}$, and using the facts that $\langle v, v'\rangle = \langle v, \overline{v}\rangle = 0$, we get that

$$\langle w', \overline{w}\rangle = (\sin^2\theta)\theta' + (\cos^2\theta)\theta' + (\cos^2\theta)\langle v, v'\rangle - (\sin^2\theta)\langle \overline{v}', v\rangle$$

Using the fact that $\langle v', \overline{v}\rangle = -\langle v, \overline{v}'\rangle$ (by differentiating the identity $\langle v, \overline{v}\rangle = 0$), we can simplify the above expression to

$$\langle w', \overline{w}\rangle = \theta' + \langle v', \overline{v}\rangle$$

Which by the definition of the algebraic value of the covariant derivative is just the fact that

$$\left[\frac{Dw}{dt}\right] = \left[\frac{Dv}{dt}\right] + \frac{d\theta}{dt}$$

What we wanted to prove. □

We are now ready to give an explicit description of the algebraic value of the covariant derivative solely in terms of the metric and in $\theta$ (which is completely determined by the metric once our unit vector fields are fixed).

**Theorem 4.** Let $\varphi(u, v)$ be an orthogonal, positively oriented parametrization of a neighborhood of our oriented regular surface $S$. Let $w(t)$ be a differentiable unit vector field along the curve $\alpha(t) = \varphi(u(t), v(t))$. Then

$$\left[\frac{Dw}{dt}\right] = \frac{1}{2\sqrt{EG}}\left(G_u\frac{dv}{dt} - E_v\frac{du}{dt}\right) + \frac{d\theta}{dt}$$

*Proof.* Letting $e_1 = \frac{\varphi_u}{\sqrt{E}}$ and $e_2 = \frac{\varphi_v}{\sqrt{G}}$ be an orthonormal frame for our neighborhood of $S$, we may use the previous lemma and write

$$\left[\frac{Dw}{dt}\right] = \left[\frac{De_1}{dt}\right] + \frac{d\theta}{dt}$$

Where $e_1(t) = e_1(u(t), v(t))$, the restriction of the vector field $e_1$ to our curve $\alpha$. By the definition of $\left[\frac{De_1}{dt}\right]$ and the fact that $N \times e_1 = e_2$,

$$\left[\frac{De_1}{dt}\right] = \langle\frac{de_1}{dt}, e_2\rangle = \langle(e_1)_u, e_2\rangle\frac{du}{dt} + \langle(e_1)_v, e_2\rangle\frac{dv}{dt}$$

Since $F = 0$, i.e. $\langle\varphi_u, \varphi_v\rangle = 0$, we have that

$$\langle\varphi_{uu}, \varphi_v\rangle = -\langle\varphi_u, \varphi_{vu}\rangle = -\langle\varphi_u, \varphi_{uv}\rangle = -\frac{1}{2}E_v$$

And from this, we immediately get from the quotient rule and $F = 0$ that

$$\langle(e_1)_u, e_2\rangle = \langle\left(\frac{\varphi_u}{\sqrt{E}}\right)_u, \frac{\varphi_v}{\sqrt{G}}\rangle = -\frac{1}{2}\frac{E_v}{\sqrt{EG}}$$

In the same way, we also have that $\langle(e_1)_u, e_2\rangle = \frac{1}{2}\frac{G_u}{\sqrt{EG}}$. Plugging this into our expression for $\left[\frac{De_1}{dt}\right]$ allows us to conclude that

$$\left[\frac{Dw}{dt}\right] = \frac{1}{2\sqrt{EG}}\left(G_u\frac{dv}{dt} - E_v\frac{du}{dt}\right) + \frac{d\theta}{dt}$$

And we are done. $\square$

With this, we have obtained an intrinsic description of both the Gaussian and geodesic curvature. As discussed before, the proof of the local Gauss-Bonnet will involve relating these two curvatures, which will be done by passing from the boundary of a geodesic triangle to its interior via Green's theorem. However, there will still be a term remaining, which we will build the tools to understand in the section below.

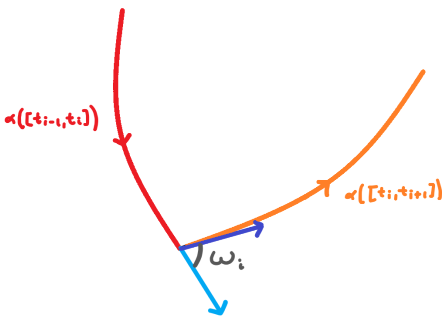
## 5 Geodesic Triangles

In this section, we will precisely define what a geodesic triangle is, and state (but not prove) a topological fact about the angles of these triangles.

**Definition 7.** A parameterized curve $\alpha : I \to S$ is a geodesic triangle if $\alpha(0) = \alpha(1)$, $\alpha$ is injective, and there exists $0 = t_0, t_1, t_2 \in [0, 1)$ such that $\alpha([0, t_1]), \alpha([t_1, t_2]), \alpha([t_2, 1])$ are smooth curves with injective differential that are geodesics of $S$. We call $\alpha(0), \alpha(t_1), \alpha(t_2)$ the vertices of our geodesic triangle.

As a notational convention, $t_3$ will always just be 1.

As we defined earlier with geodesic curvature, we define $\theta_i : [t_i, t_{i+1}] \to \mathbb{R}$ to be the function that measures the angle between $\alpha'(t)$ and $\varphi_u$ (with respect to the metric on the tangent spaces). We define the external angle $\omega_i$ at $\alpha(t_i)$ to be the difference between $\theta_i(t_i)$ and $\theta_{i+1}(t_i)$, as depicted in the picture below:




A theorem in topology tells us the following

> **Theorem 5.** (Theorem of Turning Tangents) Given a geodesic triangle (in general, any piecewise regular curve) in $S$, we have that
>
> $$\sum_{j=0}^{2}(\theta_i(t_{i+1}) - \theta_i(t_i)) + \sum_{i=0}^{2}\omega_i = 2\pi$$

This tells us that with respect to any smooth coordinate frame on a chart containing a geodesic triangle, that the change in angle of a tangent vector our geodesic triangle across each of the three edges, plus the jumps in angle at each of the three vertices, add up to $2\pi$. This is a generalization of the fact in the plane that the exterior angles of a triangle (in fact, any polygon) add to $2\pi$. The proof of this fact isn't too hard, using some basic covering space facts and continuity of the curve to prove the fact in the plane, and a short argument to pass the proof to a general surface. A detailed exposition on this fact can be found in Chapter 9-1 of [Lee18].

With this, we finally have all the tools we need to prove the local version of Gauss-Bonnet.

## 6 The Proof of Local Gauss-Bonnet

We would like to show that for a geodesic triangle with interior $\Omega$ and external angles $\omega_1, \omega_2, \omega_3$,

$$\int_{\Omega} K\, dA + \omega_1 + \omega_2 + \omega_3 = 2\pi$$

Like we discussed earlier, the crux of this proof will be relating the geodesic curvature of the boundary of the geodesic triangle (which we know), to the Gaussian curvature of the interior of the triangle. Let $\alpha : I \to S$ a piecewise smooth curve be a parameterization of the boundary of our geodesic triangle by arclength, and let $\alpha(s_0), \alpha(s_1), \alpha(s_2)$ be the vertices of the geodesic triangle. Then since the geodesic curvature on a geodesic is by definition zero (since the covariant derivative of the tangent vector field along a geodesic is zero), we have that

$$0 = \sum_{j=0}^{2}\int_{s_j}^{s_{j+1}} k_g(s)\, ds$$

We recall from Theorem 4 our intrinsic description of the covariant derivative to get that

$$\sum_{j=0}^{2}\int_{s_j}^{s_{j+1}} k_g(s) ds = \sum_{j=0}^{2}\int_{s_j}^{s_{j+1}} \frac{1}{2\sqrt{EG}}\left(G_u\frac{dv}{dt} - E_v\frac{du}{dt}\right) ds + \sum_{j=0}^{2}\int_{s_j}^{s_{j+1}} \frac{d\theta_j}{ds}\, ds \tag{5}$$

Applying Green's theorem to the first integrand gives us

$$\sum_{j=0}^{2}\int_{s_j}^{s_{j+1}} k_g(s)\, ds = \iint_{\varphi^{-1}(\Omega)} \left(\left(\frac{E_v}{2\sqrt{EG}}\right)_v + \left(\frac{G_u}{2\sqrt{EG}}\right)_u\right) du\, dv + \sum_{j=0}^{2}\int_{s_j}^{s_{j+1}} \frac{d\theta_j}{ds}\, ds$$

We recognize the first integrand as being expression of Gaussian curvature in terms of the metric (as in Theorem 3) multiplied by $-\sqrt{EG}$, so we have that

$$\iint_{\varphi^{-1}(\Omega)} \left( \left( \frac{E_v}{2\sqrt{EG}} \right)_v + \left( \frac{G_u}{2\sqrt{EG}} \right)_u \right) du\, dv = -\iint_{\varphi^{-1}(\Omega)} K\sqrt{EG}\, du\, dv = -\int_\Omega K\, dA$$

The remaining part of (5), $\sum_{j=0}^{2} \int_{s_j}^{s_{j+1}} \frac{d\theta_j}{ds}\, ds$, can be simplified using the Fundamental Theorem of Calculus and the Theorem of Turning Tangents:

$$\sum_{j=0}^{2} \int_{s_j}^{s_{j+1}} \frac{d\theta_j}{ds}\, ds = \sum_{j=0}^{2} \theta_i(s_{j+1}) - \theta_i(s_j) = 2\pi - \omega_1 - \omega_2 - \omega_3$$

Putting this all together, we have that

$$0 = \sum_{j=0}^{2} \int_{s_j}^{s_{j+1}} k_g(s)\, ds = -\int_\Omega K\, dA + 2\pi - \omega_1 - \omega_2 - \omega_3$$

And therefore,

$$\int_\Omega K\, dA + \omega_1 + \omega_2 + \omega_3 = 2\pi$$

Concluding the proof of Local Gauss-Bonnet.

# 7 Going Local to Global

The reason we worked with geodesic triangles in particular was because it turns out that all smooth Riemannian 2-manifolds can be triangularized with geodesic triangles. A proof of this is not too hard (see, for example, Exercise 9-5 in [Lee18]), but we will take it for granted.

The Euler characteristic of a surface $S$ with respect to a given triangulation is given by

$$\chi(M) = V - E + F$$

Where $V$ is the number of vertices, $E$ the number of edges, and $F$ the number of faces. The fact that this quantity is well-defined, i.e. that it doesn't depend on a given triangulation, is easily proved using CW homology. Assume $S$ is connected, since otherwise we can run this procedure on each connected component. Then applying Local Gauss-Bonnet to each triangle gives us

$$\sum_{i=1}^{F} \int_{\Omega_i} K\, dA + \sum_{i=1}^{F} \sum_{j=1}^{3} \omega_{ij} = \sum_{i=1}^{F} 2\pi$$

We'll define the interior angles $\theta_{ij}$ of the geodesic triangle $\Omega_i$ as $\pi - \omega_{ij}$, so we may write the above expression as

$$\sum_{i=1}^{F} \int_{\Omega_i} K\, dA + 3\pi F - \sum_{i=1}^{F} \sum_{j=1}^{3} \theta_{ij} = 2\pi F$$

Each interior angle $\theta_{ij}$ lies in a unique triangle, and the sum of all interior angles around a vertex is exactly $2\pi$, so $\sum_{i=1}^{F} \sum_{j=1}^{3} \theta_{ij}$ becomes $2\pi V$ and we can write our expression as

$$\sum_{i=1}^{F} \int_{\Omega_i} K\, dA = 2\pi V - \pi F$$

Since each edge appears in exactly two triangles, we can count the number of edges by counting three times the number of faces (there are three edges in a triangle), and dividing by 2 (each edge is double counted), so $2E = 3F$. Therefore, $F = 2E - 2F$ and we finally get that

$$\sum_{i=1}^{F} \int_{\Omega_i} K\, dA = 2\pi V - 2\pi E + 2\pi F = 2\pi\chi(M)$$

And this completes the proof of Gauss Bonnet!

## References

[Car76] Manfredo do Carmo. *Differential Geometry of Curves and Surfaces*. Prentice-Hall, 1976.

[Lee18] John Lee. *Introduction to Riemannian Manifolds*. Springer International Publishing, 2018.